Vol. 1 . 1

A

Rudimentary & Elementary Treatise

ON

DESCRIPTIVE GEOMETRY.

BY

J. F. HEATHER, M.A.

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JOHN WEALE.

AN

ELEMENTARY TREATISE

ON

DESCRIPTIVE GEOMETRY,

WITH A

THEORY OF SHADOWS AND OF PERSPECTIVE:

EXTRACTED FROM THE FRENCH OF G. MONGE.

TO WHICH IS ADDED,

A DESCRIPTION OF THE

PRINCIPLES AND PRACTICE OF ISOMETRICAL PROJECTION.

THE WHOLE BEING INTENDED AS AN INTRODUCTION TO THE APPLICATION OF DESCRIPTIVE GEOMETRY TO VARIOUS BRANCHES OF THE ARTS.

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PREFACE.

Having been solicited by the Publisher of the Rudimentary Treatises to supply a series of works upon the application of Descriptive Geometry to various branches of the useful arts, it has been deemed advisable to precede them by a treatise on the Elements of Descriptive Geometry itself. This appeared the more necessary from the poverty of our scientific literature in works upon this elegant branch of study, which, from its many and important practical applications, appears to demand a far more prominent place in a system of elementary education than has hitherto been accorded to it in this country.

To supply this want we have had recourse to the original work of G. Monge, both because, having been the first to reduce it to a system, he must be regarded as the inventor of the art, and because that work was originally intended to occupy the very position here given to it as a prelude to a series of works upon the application of Descriptive Geometry to the useful arts. A considerable portion of Monge's work does, however, avowedly go beyond this purpose, and, being limited in the extent and price of these treatises, this portion has been altogether omitted.

The applications of Descriptive Geometry to the constructions of perspective, and to the exact determinations of the outlines of shadows, are remarkable for their generality and simplicity. The papers from which the theories of these subjects here given have been translated, were compiled from Monge's unpublished lectures, by Mons. Brisson, Ingénieur des Ponts et Chaussées, the editor of the latest editions of Monge's work.

The present volume concludes with a brief account of the principles and practice of Isometric Projection, an elegant system due to the inventive genius of the late Professor Farish, formerly Jacksonian Professor of Natural and Experimental Philosophy in the University of Cambridge, who was thus enabled to give such pictures of machinery as should enable a person to construct them without any other assistance.

The method of Horizontal Projection, in which objects are described by their projections upon one plane, considered as horizontal, and by representing numerically the altitudes above the plane of the several parts of the projection, being not generally applicable with advantage as a means of graphical delineation, will be discussed in connection with that branch of the arts (surveying and mapping) in which it is most advantageously employed.

ROYAL MILITARY ACADEMY,
14th July, 1851.

CONTENTS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

DESCRIPTIVE GEOMETRY.

CHAPTER I.

THE OBJECT OF DESCRIPTIVE GEOMETRY.

1. DESCRIPTIVE geometry has two objects in view: the first to determine methods for representing on a sheet of paper which has only two dimensions, viz., length and breadth, the forms of solid bodies which have three, length, breadth, and thickness, provided that these bodies can be rigorously defined.

The second object is to determine methods for recognising from an exact construction, the forms of solid bodies, and thence to deduce all the truths resulting both from their forms and respective positions.

We shall first point out the steps which long experience has discovered to be best adapted to accomplish the first of these objects; and afterwards proceed to the second.

CONSIDERATIONS FOR DETERMINING THE POSITION OF A POINT IN SPACE. THE METHOD OF PROJECTIONS.

2. Since the surfaces of all bodies may be considered as composed of points, the first step that we intend taking in this matter will be to shew how to represent the position of a point in space.

Space is unlimited; all its parts are exactly alike, not having

anything which distinguishes one from another, and therefore parts of space cannot be employed as terms of comparison to fix the position of a point.

In order, then, to define the position of a point in space we must refer it to other objects, considered as quite distinct from the surrounding space, whose positions are known as well to us as to the person to whom we wish to point them out; and this is found to be best accomplished by selecting for the objects of comparison, planes whose positions are easily imagined.

3. Definition.—The projection of a point upon a plane is the foot of the perpendicular let fall from the point upon the plane.

If then we have two planes whose positions in space are known, and on each of these planes take the projections of a point whose position we wish to describe, this point will be completely determined.

In fact, if through the projection in the first plane we suppose a perpendicular to this plane to be drawn, it is evident that this line will pass through the point we are describing; and in the same way, if through the projection in the second plane we suppose a perpendicular to this second plane to be drawn, it will pass through the same point. Consequently, this point being at the same time in two straight lines, whose positions in space are known, will be the single point of their intersection, and is thus completely determined.

In the following articles we shall point out the methods for rendering this process easy in practice, and of such a nature that it may be employed on a single sheet of paper.

4. If from every point of an indefinite straight line A B (pl. I., fig. 1), placed in any manner in space, we imagine perpendiculars to be let fall upon a plane L M N O, whose position is given, then all the points in which these perpendiculars meet the plane will form another indefinite straight line *ab*; for they will be all included in the plane drawn through A B perpendicularly to the plane L M N O, and can only meet this last at the

common intersection of the two planes, which is known to be a straight line.

The straight line ab, which passes through the projections upon the plane L M N O of all the points of the straight line A B, is called the projection of the straight line A B on this plane.

Since two points are sufficient for determining the position of a straight line, in order to construct the projection of a straight line, it is only necessary to construct the projections of two of its points, and the straight line drawn through these two projections will be the projection required.

Hence, if the proposed straight line is itself perpendicular to the plane of projection, its projection will be reduced to the single point in which it meets the plane.

If we know the projections ab, $a'b'$, on two planes, L M N O, L M P Q (pl. I., fig. 2), which are not parallel to each other, of an indefinite straight line A B, this line is determined: for if through $a\,b$, one of its projections, we suppose a plane drawn perpendicular to L M N O, this plane, known in position, will necessarily pass through the straight line A B; and in the same way, if through the other projection, $a'\,b'$, we suppose a plane perpendicular to L M P Q, this plane, known in position, will also pass through the straight line A B. The position of this straight line, which is common to the two planes, must be then at their common intersection, and will therefore be completely determined.

5. The methods described in the two last articles are independent of the position of the planes of projection, and apply equally well whatever may be the angle at which these two planes are inclined to each other. But if this angle be very obtuse, then the angle formed by the two planes which are perpendicular to the planes of projection will be very acute; and in practice, small errors might introduce errors of considerable importance in determining the actual positions of the points and lines. To be as exact as possible, unless by so doing the construction of a problem be rendered more difficult, the planes of projection are

selected at right angles to each other. Moreover, since the artists who make use of the method of projections are usually familiar with the position of a horizontal plane and the direction of a plumb line, it is usual to suppose that of the two planes of projection, one is horizontal, and the other vertical.

6. The necessity for making the two projections upon one sheet of paper in designs, and on the same area in operations at large, has determined artists to suppose the vertical plane to be turned about its intersection with the horizontal plane, as upon a hinge, and fall upon the horizontal plane, so as to form with it but one and the same plane, and to construct projections on it in this position.

So that in fact the vertical projection is always traced upon a horizontal plane, and it must be always supposed to be erected and restored to its vertical position by being turned about the intersection of the vertical and horizontal planes through a quarter of a complete revolution. For this purpose the intersection must be traced in the drawing in a very clear manner.

Thus, in fig. 2, the projection $a' b'$ of the straight line A B is not drawn on a plane which is really vertical; but we imagine that this plane is turned about the straight line L M until it coincides with L M P′ Q′, and that the vertical projection $a' b'$ is drawn upon it when in this position.

This arrangement, besides other advantages, lessens the work of projecting. For suppose that the points $a\, a'$, are the horizontal and vertical projections of a point A; the plane which passes through the straight lines A a, A a', will be perpendicular to each of the two planes of projection, because it passes through straight lines which are perpendicular to them, and will therefore be perpendicular to their common intersection L M; and the straight lines a C, a' C, which are the intersections of these two planes of projection with that plane, will be themselves perpendicular to L M.

Now whilst the vertical plane is revolving about L M as a

hinge, the straight line a' C continues to be at right angles to L M, and remains so when, this plane having come into a horizontal position, it coincides with the straight line C a''. Then the two lines a C, C a'', both passing through the point C, and being both perpendicular to L M, must be in one and the same straight line; and the same remark will apply to the straight lines b D, D b'', having the same relation to any other point B. Whence it appears that, if the horizontal projection of a point is known, its projection upon the vertical plane, supposed to have revolved into the position just described, will be in the production of the straight line drawn through the horizontal projection perpendicular to L M, the common intersections of the two planes, and *vice versa*. This result is applied constantly in practice.

As yet the straight line A B has been considered of indefinite length, and its direction only has been considered; but, if we regard it as being limited by two of its points A, B, it may be necessary to know its length, which we proceed to determine by means of its two projections.

When a straight line is parallel to one of the two planes of projection, its length is equal to its projection on this plane; for the line and its projection being both terminated by two perpendiculars to the plane of projection, are parallel to each other, and joined at their extremities by parallel lines. In this case then, when the projection is given, the length of the straight line, which is equal to it, is also given. It is evident that a straight line is parallel to one of the two planes when its projection on the other is parallel to their intersection.

If a straight line is oblique to each of the two planes, then its length is greater than either of its projections; but may be deduced from them very simply.

Let A B (pl. I., fig. 2) be a straight line whose two projections $a\,b$, $a'\,b'$, are given and whose length is required. If we suppose a horizontal line A E to be drawn through one of its extremities A, in the vertical plane passing through A B, and produced until it meets the vertical let fall from the other

extremity in the point É, a right angled triangle, A E B, will be formed, which is to be constructed in order to obtain the length of the straight line A B, which is its hypotheneuse. Now besides knowing the angle at E to be a right angle, we also know that A E is equal in length to the given projection $a\,b$. Also drawing in the vertical plane through a' a horizontal line $a'e$, it cuts the vertical b' D at the point e, which is the projection of the point E; and therefore $b'\,e$ is the vertical projection of B E, and of the same length with it. Hence, having determined the lengths of the two sides containing the right angle, there will be no difficulty in constructing the hypotheneuse A B of the triangle.

Fig. 2 is drawn in perspective, and is quite distinct from the constructions employed in projections: we are now going to construct the figure for this first question in its simplest form.

Taking the straight line L M (pl. I., fig. 3) as the intersection of the two planes of projection, let the straight lines $a\,b$, $a''\,b''$, be the given projections of a straight line whose length is required. Draw through a'' an indefinite straight line H e meeting bb'' at the point e, and from e measure a distance, e H, in the direction $e\,a''$, equal to $a\,b$; then H b'', the hypotheneuse of the right angled triangle H $e\,b''$, will be the length required.

Since the two planes of projection are at right angles to each other, the same result might have been obtained by making the construction on the horizontal plane.

It appears then that the dimensions of bodies which are included by plane superficies, by rectilinear edges, and by solid angles, may be found, when their projections on two planes are known; for an edge of such a body must either be parallel to one of the planes of projection, or oblique to each of them; and in the first case the required length will be equal to that of its projection, and in the second, it may be deduced from the two projections by the method just investigated.

8. There is no general rule for constructing the projections of rectilinear solid bodies. In the same way as for analytical investigations, the process to be adopted in each particular case

must depend upon the conditions given; and practice alone can enable us to pursue the best methods. But, again, as in algebra, when a problem has been expressed by equations, there are rules for their solution; so also in descriptive geometry, when the projections have been constructed, there are general methods for constructing all the results which proceed from the forms and respective positions of the bodies.

ON THE CONVENTION BEST ADAPTED TO EXPRESS THE FORMS AND POSITIONS OF SURFACES. ITS APPLICATION TO THE PLANE.

9. The convention employed as the base of the method of projections is well adapted to express the position of a point in space, and of a straight line unlimited or limited, and, consequently, for representing the form and position of a body bounded by plane faces, by rectilinear edges, and by solid angles; because in this case a body is completely known when we know the position of all its edges and of the vertices of all its angles. But should the body be bounded either by one curved surface having all its points subject to the same law, as in the case of a sphere, or by the several parts of different curved surfaces, as in the case of a body whose surface has been formed in a lathe, this convention will not only be inconvenient and impracticable, but will be insufficient for accomplishing the object proposed, and wanting in fertility.

In the first place it is evident that this convention when taken singly would be inconvenient and even impracticable; for, in order to express the position of all the points of a curved surface, not only must each be distinguished by its horizontal and vertical projections; but these projections must be so connected that the horizontal projection of one point shall not be mistaken for that which corresponds to the vertical projection of another; and, as the most simple manner of connecting these two projections is to join them by a straight line perpendicular to the line of intersection of the planes of

projection, by the application of this method a drawing would be crowded with a prodigious number of lines; and the more exact we might wish to draw the figure, the more confusion would exist.

We proceed to show that this method would be insufficient, and wanting in fertility.

Among the infinite number of curved surfaces, some are limited in all directions, and consequently their projections must be limited in extent. The sphere for instance is such a surface; the extent of its projection on a plane is limited to that of a circle of the same radius as the sphere; and the plane upon which this projection is made can be conceived to be sufficiently large to contain it. But all cylindrical surfaces are indefinite in some one direction, similarly to the straight lines which generate them. Even "a plane" which is the most simple of surfaces is indefinite in two directions. There are also a great number of surfaces, whose numerous sheets extend at the same time into all parts of space; and, since the planes upon which the projections are drawn are necessarily of a limited size, if the only method for knowing the nature of a curved surface was by the two projections of each of the points through which it passes, this would be applicable only to the points of the surface which correspond to the extent of the planes of projection, whilst all those which are beyond these planes could neither be expressed nor known; and therefore this method is insufficient. Lastly, it is wanting in fertility; for by it we cannot deduce anything relating to the tangent planes to the surface, to its normals, to its double curvature at each point, to its lines of inflexion, to its re-entering edges, to its multiple lines, to its multiple points, and, in a word, to all those affections which must be considered, to attain a complete knowledge of a curved surface.

It is necessary, therefore, to have recourse to an additional convention, which shall be consistent with the first, and shall at the same time supply that in which it is deficient. This new convention is the subject of the next article.

10. Every curved surface may be regarded as generated by the movement of a line, either constant in form while it changes its position, or variable at the same time both in form and in position in space; a proposition which, since it may not be so easily understood in this general form, we shall explain by taking a few examples with which we are already familiar.

Cylindrical surfaces may be generated in two principal ways, either by the movement of a straight line which always remains parallel to a given straight line, while it moves so as to pass through all the points of a given curve, or by the movement of the curve which in the first method served as a guide to the straight line, in such a way that, one of its points moving along a given straight line, all its other points may describe lines parallel to this straight line. In both of these methods the generating line, which in the first case was a straight line, and in the second was a curve, is constant in form and does but change its position in space.

Conical surfaces may also be generated in two principal ways. We may first regard them as generated by an indefinite straight line, which always passes through a given point, and moves so that it rests constantly upon a given curve which directs it in its movement. The single point through which the straight line always passes is the centre of the surface, the name of vertex, by which it is sometimes designated, being incorrectly applied. In this mode of generation the generating line remains constant in form, never ceasing to be a straight line.

In considering the second method in which conical surfaces may be generated we shall for the sake of simplicity only refer to the case of such as have circular bases.

These surfaces may be regarded as swept out by the circumference of a circle, which moves so that its plane is always parallel to itself, and its centre on a straight line which passes through the vertex of the cone, while it expands or contracts, so that its radius during the motion is constantly proportioned to

the distance of its centre from the vertex. We observe that if in its movement the plane of the circle tends to approach the vertex of the surface, the radius of the circle decreases so as to become nothing when the plane passes through the vertex, and that this radius changes direction and increases indefinitely, as the plane, after having passed through the vertex, extends itself more and more. In this second mode of generation, not only does the circumference of the circle, which is the generating curve, change its position; but it also changes its form at each instant during its motion, because its radius changes, and consequently its curvature and size.

To take one more example:—

A surface of revolution may be generated by the movement of a plane curve, which revolves about a right line situated anywhere in its plane. In this way of regarding it the generating curve is constant in form, and variable in position only.

But we may also regard it as generated by the circumference of a circle which moves so that its centre is always in the axis, and its plane perpendicular to this axis, and which expands or contracts, so that its radius at each instant is equal to the distance between the two points in which the plane of the circle cuts the axis, and a curve given in space. The generating curve in this way changes at the same time both its form and position.

These three examples are sufficient to make it evident that all curved surfaces may be generated by the movement of certain curved lines, and that there is not any surface whose form and position cannot be completely determined by an exact and complete definition of its mode of generation. This new consideration constitutes the complement to the method of projections, and we shall frequently have occasion, as we proceed, to observe its simplicity and fertility.

The form and position of a curved surface is not, then, determined by knowing the projections of the particular points through which it passes, but by being able to construct the generating curve through any point according to the form and

position which belong to it in passing through this point: and here it should be observed that—firstly, each curved surface may be generated in an infinite number of different ways, and that the ingenuity and skill of the constructor are shown in selecting that which employs the most simple curve, and requires considerations the most easily understood; and secondly, that experience has taught us that instead of regarding a curved surface as generated by one curve in one particular manner, which requires a knowledge of the law of the curve's motion, and also of its change of form, it is generally preferable to consider two different methods of generating the surface at the same time, and point out the construction for the two generating curves through each point.

Therefore, in order by descriptive geometry to express the form and position of a curved surface, it is sufficient to give for any point whatever of this surface, whose projections may be taken at pleasure, the method for constructing the horizontal and vertical projections of two different generating curves which pass through this point.

11. Let us now apply these general considerations to "a plane," which of all surfaces is the most simple, and the most frequently employed in geometrical constructions.

A plane may be regarded as generated by a straight line whose primary position is given, and which moves in such a manner that each of its points describes a straight line parallel to a second given straight line: if this second straight line is situated in the plane we are considering, we may also say that the plane is generated by the second straight line moving so that all its points describe straight lines parallel to the first.

A good idea of the position of a plane is thus afforded us by the consideration of two straight lines, either of which may be looked upon as generating it; and the position of these lines in the plane which they generate is perfectly indifferent. It is only requisite then, for the method of projections, to select those lines which require the simplest constructions; and for this reason, in

descriptive geometry, the position of a plane is indicated by the two straight lines in which it cuts the planes of projection.

Definition.—The two straight lines, which are the intersections of any plane, with the planes of projection, are called the traces of that plane.

It is evident that these traces must meet the common intersection of the two planes of projection at the same point, and this, therefore, is the point in which they meet each other.

THE SOLUTIONS OF SOME ELEMENTARY QUESTIONS RESPECTING THE STRAIGHT LINE AND THE PLANE.

12. Prob. 1 (pl. I., fig. 4). Having given a point whose projections are D, *d*, and a straight line whose projections are A B and *a b*, to construct the projections of a second straight line which shall pass through the given point and be parallel to the first straight line.

Solution.—The horizontal projections of the given straight line and of the required straight line, are parallel to each other; for they are the intersections of two parallel vertical planes with the same plane; and for a like reason the vertical projections of these two lines are also parallel. Moreover, the required straight line must pass through the given point, and therefore its projections must respectively pass through those of this point. Therefore, E F drawn through D parallel to A B, and *ef* drawn through *d* parallel to *ab*, are the projections required.

13. Prob. 2 (pl. I., fig. 5). Having a plane given whose traces are A B and B C, and a point whose projections are G, *g*, to construct the traces of a second plane drawn through the given point parallel to the given plane.

Solution.—The traces of the required plane and of the given plane must be respectively parallel to each other; because, considering either the two horizontal or the two vertical traces, they are the intersections of two parallel planes with one and the same plane. It only remains then to find for each trace

a point through which it passes. For this purpose suppose a horizontal line to be drawn in the required plane through the given point: this line will be parallel to the trace A B, and meet the vertical plane of projection at some point in the vertical trace of the required plane; and its two projections will be obtained (12) by drawing through g a straight line g F, parallel to the axis L M, which is the vertical projection of the trace A B, and through G a straight line G I parallel to A B. If then G I is produced until it meets L M at the point I, this point will be the horizontal projection of the point in which the horizontal line meets the vertical plane, which latter point must therefore be found somewhere in the vertical I F drawn through I; but it is also situated in g F, and must therefore coincide with F, the common intersection of g F and I F. If then a straight line is drawn through F parallel to B C, it will be the vertical trace of the required plane; and if, after having produced this trace until it meets L M in E, we draw through E a line parallel to A B, we shall have the horizontal trace of the same plane.

If instead of conceiving a horizontal line in the required plane, we had imagined a line drawn parallel to the vertical plane, we should have been led to the following construction.

Draw through G the indefinite straight line G D parallel to L M; and through g draw g H parallel to C B, and produce it until it cuts L M at the point H, through which draw H D perpendicular to L M, cutting G D at the point D. The straight line drawn through D parallel to A B is the required horizontal trace; and F E drawn parallel to B C through the point E, where the horizontal trace meets L M, is the vertical trace.

14. Prob. 3 (pl. I., fig. 6). Having given a plane whose traces are A B, B C, and a point whose projections are D, d, to construct the projections of a straight line drawn from the point perpendicular to the plane, and also the projections of the point of intersection of this line with the plane.

Solution.—The perpendiculars D G, d g, let fall from the

points D, d, on the respective traces of the plane, will be the indefinite projections of the required line; for were we to suppose a vertical plane to pass through this line, this plane would intersect the horizontal plane and the given plane in two straight lines which would be both perpendicular to A B, the common intersection of these two planes: but the first of these lines being the horizontal projection of the vertical plane, is also the projection of the perpendicular included in it; and therefore the horizontal projection of this perpendicular will pass through D, and be perpendicular to A B; and in like manner it may be shown that its vertical projection will pass through d, and be perpendicular to B C.

With regard to the point in which the perpendicular meets the plane, it is evident that it lies in the intersection of the vertical plane drawn through the perpendicular, with the given plane, of which intersection the horizontal projection is E F. If we had the vertical projection, $f\,e$, of this intersection, it would contain the vertical projection of the required point, and since this vertical projection is also in $d\,g$, it would coincide with g, the intersection of $f\,e$ and $d\,g$. It only remains then to draw $e\,f$. Now the intersection of the given plane with the vertical plane which is perpendicular to it, meets the horizontal plane at the point E, the vertical projection of which is e, found by drawing E e perpendicular to L M; and it meets the vertical plane of projection at a point whose horizontal projection is F, the intersection of L M with D G, produced if necessary, and whose vertical projection is f, the intersection of the vertical F f and the trace B C. Having found g, the vertical projection of the required point, it is easy to construct its horizontal projection; for the indefinite straight line G g drawn perpendicular to L M, will contain this projection, and, since this projection is also in D G, it will be at the intersection G of these two lines.

15. PROB. 4 (pl. I., fig. 7). Having given a straight line whose projections are A B, $a\,b$, and a point whose projections

are D and *d*, to construct the traces of a plane drawn through the point perpendicular to the straight line.

Solution.—We already know from the preceding problem that the traces of the required plane must be respectively perpendicular to the projections of the given line, and it only remains to find for each trace a point through which it must pass. For this purpose suppose a horizontal line to be drawn in the required plane through the given point, and to be produced until it meets the vertical plane of projection; the indefinite horizontal straight line *d* G will be the vertical projection of this line, while its horizontal projection will be found by drawing D H through D perpendicular to A B. The point H, then, in which D H meets the axis L M, is the horizontal projection of the point where the horizontal line through the given point meets the vertical plane of projection; so that this point of meeting, being both in the vertical H G and in the horizontal line, *d* G, is at the intersection, G, of these two lines, which is therefore one of the points through which the vertical trace passes. We obtain this trace, then, by drawing through G a straight line F C perpendicular to *a b;* and thence by drawing through C, where the vertical trace meets L M, a straight line C E perpendicular to A B, we shall have the horizontal trace of the required plane.

Instead of supposing a horizontal line to be drawn through the given point, we may suppose a line drawn parallel to the vertical plane, which by a similar reasoning will give us the following construction. Draw through D a line parallel to L M and through *d* a line perpendicular to *a b*; from the point where this last line meets L M draw a perpendicular to L M in the horizontal plane; and the intersection of this perpendicular with the line drawn through D parallel to L M, is a point through which the horizontal trace of the required plane passes.

If it were required to draw a perpendicular from the given point upon the given straight line, we should construct the projections of the point where the straight line intersects the plane

whose traces are C E, C F; and, as we know D, *d*, we should have then two known points in each of the planes of projection through which the projections of the perpendicular passes.

16. Prob. 5 (pl. II., fig. 8). Two planes being given in position whose traces are A B, *a b*, and C D, *c d*, respectively, to construct the projection of the straight line in which they intersect.

Solution.—Since every point of the trace A B is situated in the first of these two planes, and every point of the trace C D is situated in the second, the point E, in which these traces intersect, is common to both planes, and must be a point in the required line. Similarly F, the intersection of the vertical traces A *b* and C *d*, is also a point in the required line; and, consequently, the intersection of the two planes meets the horizontal plane in E, and the vertical in F.

If, then, we take the horizontal projection of F, which is effected by drawing F *f*, perpendicular to L M, meeting L M at the point *f*, and join E *f*, we obtain the horizontal projection of the intersection of the two planes; and in like manner we obtain the vertical projection by drawing E *e* perpendicular to L M, and joining F *e*.*

17. Prob. 6 (pl. II., fig. 9). Two planes being given whose traces are A B, A *b*, and C D, C *d*, respectively, to construct the angle between them.

Solution.—After having constructed, as in the preceding question, the horizontal projection of the intersection of the two planes, suppose a third plane perpendicular to each of them, and consequently perpendicular to their common intersection: then this third plane will intersect the two given planes in two right lines which will contain an angle equal to the required angle.

Moreover the horizontal trace of this third plane will be perpendicular to E *f*, the horizontal projection of the intersection

* Hence is seen how to construct the projections of a straight line when its traces, that is, the points in which it meets the planes of projection, are given.

of the two given planes, and will form with the two other right lines a triangle, in which the angle opposite to the horizontal side will be the required angle. We have then merely to construct this triangle.

Since it is immaterial through what point in the intersection of the two first planes, the third passes, we can take its horizontal trace where we please in the horizontal plane, provided that it is perpendicular to E *f*. Let any straight line G H be drawn perpendicular to E *f* cutting E *f* at the point I, and meeting E C and E A at the points H and G respectively: this straight line will be the base of the triangle which we are desirous of constructing. Suppose now that the plane of this triangle revolves about G H, as upon a hinge, until it coincides with the horizontal plane. During this movement, the vertex of the triangle always remains in the vertical plane passing through *f* E; because this plane is perpendicular to G H; and, therefore, when the plane of this triangle comes into coincidence with the horizontal plane, its vertex must lie somewhere in E *f*; and we have only further to determine the altitude of the triangle, or the length of the perpendicular let fall from I on the intersection of the two given planes. Now this perpendicular being situated in the vertical plane, drawn through E *f*, were we to suppose the plane of the triangle E *f* F to turn about the vertical F *f*, until it coincides with the vertical plane of projection, *f* E, *f* I would come into the positions *f e*, *f i*. The straight line F *e*, then, is the length of the line of intersection comprised between the two planes of projection; and, if from the point *i* a perpendicular to F *e* is drawn, this perpendicular is the altitude required.

Measure off then from I E a line I K equal to *i k*, and complete the triangle G K H; and the angle contained by K G and K H will be equal to the angle between the two given planes.

18. Prob. 7 (pl. II., fig. 10). Two straight lines which intersect each other in space being given by means of their horizontal and vertical projections A B, A C, and *a b*, *a c*, to construct the angle between them.

Before proceeding to the solution we remark that, since the two given straight lines are supposed to intersect, the points A and *a*, in which the horizontal and vertical projections, respectively, meet each other, will be the projections of their point of intersection, and must, therefore, be situated in the same straight line *a* G A perpendicular to L M. If A and *a* were not in the same straight line at right angles to L M, the given straight lines would not intersect, and, consequently, would not be situated in one and the same plane.

SOLUTION.—Suppose the two straight lines to be produced until they meet the horizontal plane in two points, or horizontal traces, and let us first construct these traces. For this purpose produce *a b*, *a c*, until they meet L M at the points *d* and *e*, which will be the vertical projections of the required traces; and therefore the traces themselves will coincide with the points D and E, in which perpendiculars to L M, drawn through *d* and *e* in the horizontal plane, meet the horizontal projections of the two given lines.*

Join D E. Then the straight line D E and the two parts of the given straight lines, included between their point of intersection and the points D, E, will form a triangle, of which the angle opposite to D E will be the angle required; and consequently, the question is reduced to that of constructing this triangle. Letting fall, then, from A on D E the indefinite perpendicular A F, if we imagine the plane of the triangle to revolve about its base D E, as upon a hinge, until it coincides with the horizontal plane, the vertex of the triangle during this movement will always remain in the vertical plane passing through F A, and at last will be situated in F A produced at some point H, whose distance from D E we proceed to determine.

Now, since A F is the horizontal projection of this distance, and the perpendicular height of one end of this distance above

* Similarly the vertical trace of the line A C, *a c*, would be found by producing A C to meet L M in the point I; then drawing through this point a perpendicular to L M its intersection *i* with the vertical projection *a c* will be the trace required.

the other is a G, if from G we set off upon L M a distance Gf equal to A F and join af, this hypotheneuse (7) will be the distance required. If, then, we measure off F H equal to af and join H E, H D, the angle D H E shall be the angle required.

19. PROB. 8. Having given the projections of a straight line, and the traces of a plane, to construct the angle between the straight line and the plane.

SOLUTION.—If through a point in the given straight line we suppose a perpendicular drawn to the given plane, the angle which this perpendicular will form with the given straight line will be the complement of the required angle.

Now by taking in the two projections of the given straight line two points both situated on any one perpendicular to the intersection of the planes of projection, and through them drawing perpendiculars to the respective traces of the given plane, we shall obtain the horizontal and vertical projections of this second straight line; and the question will be reduced to that of constructing the angle included by two straight lines which intersect, and will depend upon the preceding case (18).

20. When we wish to make a map of a country, we usually suppose that all the remarkable points in it are connected together by straight lines which form triangles, and similar triangles on a smaller scale, are to be described on the map in the same order as those they represent. The operations to be performed upon the ground consist chiefly in measuring the angles of these triangles; and, in order that these angles may be described at once on the map, they should be all situated in a horizontal plane, parallel to that of the map. If the plane of the angle is oblique to the horizon, we have to construct the horizontal projection of it, and not the angle itself; and it is always possible to find this projection when, after having measured the angle, we have also measured the angles which its two sides make with the horizon. The operation for performing this is known by the name of the reduction of an angle to the horizon.

PROB. 9. To reduce a given angle to the horizon having given the angles which its sides make with the horizon.

SOLUTION.—Let A (pl. II., fig. 11) be the horizontal projection of the vertex of the required angle, and A B that of one of its sides, so that it remains to construct A E, that of the other side. Suppose the vertical plane of projection to pass through A B; and, having drawn through A an indefinite vertical line A *a*, take any point in it, *d*, to be regarded as the vertical projection of the vertex of the angle observed. This being done, if through the point *d* we draw a straight line *d* B, making with the horizontal line an angle *d* B A equal to that which the first side makes with the horizon, the point B will be the place in which this side meets the horizontal plane. In the same manner, if through the point *d* we draw the straight line *d* C, making with the horizontal line an angle equal to that which the second side makes with the horizon, and if from the point A as a center, with radius equal to A C, we describe the indefinite arc of a circle C E F, the second side can meet the horizontal plane only at some point in the arc C E F. There only remains then to find the distance of this point from some other point as B. Since this last distance is in the plane of the observed angle, were we to draw a straight line *d* D, so that the angle D *d* B may be equal to the observed angle, and then set off *d* D equal to *d* C, and join D B, D B will be equal to the distance sought.

Then if from point B as a center, with radius equal to B D, we describe an arc of a circle, the point E in which it cuts the first arc C E F, will be the point in which the second side meets the horizontal plane; and therefore the straight line A E will be the horizontal projection of this side, and the angle B A E, that of the observed angle.

CHAPTER II.

ON TANGENT PLANES, AND NORMALS TO CURVED SURFACES.

21. Every curved surface may be regarded as generated in several different ways by the movement of curved lines. If then we suppose two different generating curves to pass through any point whatsoever in the surface, and to retain the same positions as in generating the surface they would occupy when passing through this point, and if we imagine tangents to each of these generating curves at the same point, the plane drawn through these two tangents is the tangent plane. The point in which the two generating curves intersect in the surface, and which is, at the same time, common to the two tangents and to the tangent plane, is the point of contact of the surface and the plane.

22. Definition.—The straight line drawn through the point of contact at right angles to the tangent plane is called a normal to the surface. It is perpendicular to an element of the surface at this point, because the direction of this element coincides on every side with that of the tangent plane, which may be looked upon as an extension of the element.

The knowledge of tangent planes, and of normals to curved surfaces is very useful in many of the arts; and for some is absolutely indispensable. We will notice here a single instance of each of these cases, one taken from architecture, the other, from painting.

23. The different parts of which arches in masonry are composed, are called "voussoirs"; and the name "joint" is given to the face in which two voussoirs touch each other, whether they be situated in the same, or in two adjacent layers.

The position of the joints in arches is subject to several conditions which should be necessarily satisfied; but at present

we will confine our attention to that which relates to the matter before us.

One condition necessary to the proper position of the joints, is that they should be perpendicular to each other, and that they should both meet the surface of the arch at right angles. If this law were departed from perceptibly, not only would the proper proportions be disturbed, but the arch would be liable to be rendered less firm and less durable; for, if one of the joints were oblique to the surface of the arch, of the two voussoirs contiguous to this joint, one would have an obtuse and the other an acute angle; and in the reaction which these two voussoirs exert on each other, these two angles would not afford the same resistance; from the brittleness of the material the acute angle might give way, which would alter the form of the arch and endanger the strength of the building. Thus the divisions of an arch into voussoirs requires in fact the consideration of tangent planes, and of normals to the curved surface of the arch.

24. Let us now take an example from an art to which rigorous geometry might at first sight be thought inapplicable.

Painting is usually regarded as composed of two distinct branches. The one is "art" properly so called: it has for its object to excite in the spectator some definite emotion, to inspire in him some particular feeling, or to raise such a train of thought as will dispose him to receive a certain impression. It supposes in the artist a habit of the deepest contemplation: it requires on his part the most intimate knowledge of the nature of things, of the manner in which we are influenced by them, and of the signs, even involuntary, by which this influence is manifested: it is subject to no general rules; it will only bear suggestions.

25. The other branch of painting is, to speak properly, the mechanical; its object is the exact execution of the conceptions springing from the first branch of the art. Here nothing is arbitrary; everything can be foreseen by strict reasoning, because

everything is the necessary result of objects perfectly known, and of given conditions. When the form, and position of an object are determined; when its nature is also known, and the nature, number, and positions, of all the bodies which can cast light upon it, whether directly, or by reflection; when the position of the eye of the spectator is fixed; when in fact every circumstance which can influence vision is well known and established, the tint of each point of the visible surface of the object is completely determined. Everything which relates to the colour and brightness of this tint depends upon the position of the tangent plane at the point with respect to the illuminating bodies and the eye of the spectator; it may be found by reasoning alone; and when it is thus determined should be applied with exactness. Everything expressed with insufficient strength, everything exaggerated, would change the appearances, alter the forms, and produce an effect different from that which the artist expects.

I am quite aware that the rapidity of execution, which is often necessary, would but seldom allow the practice of a method which would deprive the mind of every material aid, and would leave it entirely dependant upon its reasoning powers alone, and that it is much easier for a painter to place the objects before him, and then observe their tints, and imitate them. But if it were a common practice to consider the positions of tangent planes, and the two curvatures of surfaces at each of their points, great assistance might be derived in replacing effects which the omission of certain circumstances has prevented us from producing, or in suppressing those which arise from circumstances improperly introduced.

Vague expressions such as meplat, chiaro oscuro, which painters use, are a constant testimony of the necessity that exists for a more exact knowledge of the art, and for more rigorous reasonings.

26. Independently of its use in the arts, the consideration of tangent planes and of normals to curved surfaces, is one of

the most fruitful methods which descriptive geometry employs for the solution of questions which it would be very difficult to solve by other means.

THE METHOD OF DRAWING TANGENT PLANES THROUGH GIVEN POINTS IN CURVED SURFACES (pls. II. and III., figs. 12 to 15).

27. The general method for determining the tangent plane to a curved surface (21) is to suppose two lines to pass through the point of contact, touching at this point two different generating curves, and then to construct the plane which passes through these tangents. In some particular cases, in order to abridge the constructions, this method is not literally adhered to.

In constructing the normal, we need only remark that it consists in drawing a straight line perpendicular to a tangent plane, an operation which has been already explained (14).

28. PROB. 10. To draw a tangent plane to a cylindrical surface, through a point in the surface, the horizontal projection of which is given.

SOLUTION.—Let A B, *a b* (pl. II., fig. 12), be the horizontal and vertical projections of the given straight line, to which the generating line of the cylindrical surface must be parallel (10); let E P D be a given curve in the horizontal plane on which the generating line constantly rests, and which may be regarded as the trace of the cylindrical surface; and, lastly, let C be the given horizontal projection of the point on the cylindrical surface, through which the tangent plane is to be drawn.

Suppose now that the generating line is in the position which it assumes in passing through the point whose horizontal projection is C; and since it is a straight line, it must be its own tangent, and will be one of the straight lines which will determine the position of the tangent plane. Moreover it will be parallel to the given straight line, and therefore its two projections will be respectively parallel to A B and *a b;* so that by drawing through C an indefinite straight line E F parallel to A B we shall obtain its horizontal projection.

To find its vertical projection suppose the generating line produced in the cylindrical surface until it meets the horizontal plane; this it can do only in a point common both to the projection E F and to the curve E D F, and therefore at the intersection of these lines, which will be found by producing E F until it meets E P D.

Here two cases occur: either the straight line E F will intersect the trace of the cylinder in one point, or in several points. We will examine these two cases separately, and will first suppose that however far E F may be produced, it will meet the curve E D P only in one point.

Since D is the trace of the generating line, the vertical projection of this line will be obtained by drawing through D a straight line D *d* at right angles to L M and through *d* a straight line *df* parallel to *a b*. Thus we have the two projections of one of the straight lines through which the required tangent plane passes. Moreover the vertical projection of the point of contact must be situated in C *c'* drawn through C at right angles to L M; and being also in *df*, it must, consequently, be at *c* the point of intersection of these two two straight lines.

If the straight line E F cuts the trace E D P in several points D, E, we should proceed to make the same constructions for each of these points as we have just described for the point D, considered as the only point of intersection; and the result would only be, that the vertical projections *df*, *ef'*, of as many generating lines, and the vertical projections *c c'*, of as many points of contact as there are points of intersection between the straight line E, F and the trace E P D. In the case of fig. 12, pl. II., the trace of the surface of the cylinder is a circle which has the property of being cut by a straight line in two points, and therefore the vertical drawn through the given point C must meet the surface in two points, first, in the point whose vertical projection is *c*, and through which the generating line passes when it rests on the point D, and secondly, in another point whose vertical projection is *c'*, and through which the generating

C

line passes when it rests on the point E. These two points, although they have the same horizontal projection, are yet quite distinct from each other, and correspond to two different tangent planes. Now, for each of the points of contact, the second line which is to determine the position of the tangent plane, must be found. Were we to follow strictly the general method, that is to consider the trace as the second generating line, it would be necessary to suppose it to pass successively through each of the points of contact, and to construct a tangent at each of these points; but in the particular case of cylindrical surfaces we may make use of a more simple consideration. In fact the tangent plane at the point C, *c*, touches the surface for the whole length of the generating straight line which passes through this point, and thus touching the surface at D, which is a point in the generating line, must pass through the tangent to the trace at the point D. By similar reasoning we shall find that the tangent plane corresponding to the point C, *c*′, must pass through the tangent to the trace at E. If, then, through the two points D, E, we describe tangents to the trace E D P, and produce them until they meet L M at the points K, G, we shall have the horizontal traces of two tangent planes.

We have now merely to find the vertical traces of the same planes; and since we have already a point K in the vertical trace of one, and a point G in the vertical trace of the other, it only remains to find one more point for each.

To effect this for the first of these tangent planes, suppose the point we are constructing to be that in which a horizontal line, drawn in the tangent plane, through the point of contact, meets the vertical plane. We shall obtain the horizontal projection of this straight line by drawing through the point C a straight line parallel to the trace D K, and producing it until it meets L M in the point I; while its vertical projection is the indefinite horizontal straight line drawn through *c*. The point in which the horizontal line meets the vertical plane will be common both to the vertical I *i* and to the horizontal line *c i*,

and will be at the point, *i*, of their intersection. If then we draw a straight line through K and *i* we shall have the vertical trace of the first tangent plane. Reasoning in a similar method for the second tangent plane, we shall find its trace on the vertical plane by drawing through C a straight line C H parallel to E G, and producing it until it meets L M at the point H, through which a perpendicular H *h* is to be drawn; then through *c′* draw a horizontal line *c′ h* to meet the vertical H *h* at the point *h*, and by drawing a straight line through G and *h* we have the trace required.

29. Prob. 11.—Having given the horizontal projection of a point in a conical surface, to draw a tangent plane to the surface through this point.

The solution of this question differs in no respect from that of the preceding, except that the generating line, instead of being always parallel to itself, always passes through the vertex of which the two projections are given. The construction of this problem is given in fig. 13, pl. II., and its enunciation we leave as an exercise for the student.

30. Prob. 12.—Having given the horizontal projection of a point in a surface of revolution about a vertical axis, to draw a tangent plane to the surface, through this point.

Solution.—Let A (pl. III. fig. 14) be the given horizontal projection of the axis, *a a′* its vertical projection, B C D E F the given generating curve, supposed to be in a plane passing through the axis, and G the given horizontal projection of the point of contact.

This being premised, suppose a vertical plane whose projection is the indefinite horizontal straight line A G, to pass through the point of contact and through the axis, and the intersection of this plane with the surface of revolution will form a curve, which will be the generating curve passing through the point of contact; and if a vertical line be conceived to be drawn through G, it will meet the generating curve and therefore the sur-

C 2

face in one or in several points which will be so many points of contact, to all of which G will be the common horizontal projection. We shall find all these points of contact, as they would appear in the plane of the generating curve, by measuring off from a to e a straight line, $a\,e$, equal to A G, and by drawing through e a straight line parallel to $a\,a'$: all the points E, C, in which this straight line cuts the curve B C D E F, will be the intersections of the generating curve with the vertical passing through the point G, and will determine the heights of as many points of contact above the horizontal plane. In order to obtain the vertical projections of these points of contact, we must draw indefinite horizontal straight lines through the points E, C, in which these projections will lie; but they must also be situated in the straight line drawn through G perpendicular to L M; and therefore the points g, g', in which this straight line intersects the horizontal straight lines will be the projections of the different points of contact.

If we suppose a section formed with the surface of revolution by a horizontal plane which passes through any one point of contact, this section, which may be regarded as a second generating curve, will be the circumference of a circle whose center will be in the axis, and whose tangent, which must be perpendicular to the extremity of the radius, will also be perpendicular to the vertical plane passing through A G, in which the radius is situated. Consequently the tangent plane, which must pass through this tangent, will be perpendicular to this same vertical plane, and will have its horizontal trace perpendicular to A G. In order, then, to obtain the trace of one of the tangent planes it only remains to find its distance from the point A; and, because, if through the points E and C we draw E I, C H, tangents to the first generating curve, and produce them until they meet L M at the points I, H, the straight lines a I, a H, will be equal to these distances, therefore by measuring off from A in the direction A G the distances A i, A h equal to a I,

a H, and through i and h drawing i Q, h P perpendicular to A G, and producing them to meet L M, we shall have the horizontal traces of all the tangent planes.

In order to find the vertical traces of the same planes, we must suppose a horizontal line to pass through each of the points of contact in the corresponding tangent plane, and to be produced until it meets the vertical plane of projection. This straight line which is nothing more or less than the tangent to the circle will determine a point of the vertical trace. Now, all these horizontal lines have the same horizontal projection, namely, the straight line G K, drawn through G at right angles to L M, and terminated by L M. If, then, we draw through the point K the indefinite straight line K k k' at right angles to L M, it will contain all the points in which the horizontal lines meet the vertical plane of projection. But these points will also be respectively situated in the horizontal lines drawn through the points E, C; and, therefore, the intersections k, k', of these horizontal lines with the vertical line K k will be each in the trace of one of the tangent planes. Thus the straight line Q k will be the vertical trace of one of these tangent planes, P k' that of another, and so on if there were a greater number.

We shall for the present confine ourselves to the three preceding examples, as being sufficient to explain the method for all the surfaces whose generation we have as yet defined. In the sequel of this work, as other classes of surfaces present themselves, we shall apply the same method for the determination of their tangent planes and normals. We proceed now to propose a question, in the solution of which, the consideration of a tangent plane can be advantageously employed.

31. Prob. 13.—Two straight lines being given (pl. III. fig. 15) by their horizontal and vertical projections, A B, C D, and $a\,b$, $c\,d$, respectively, it is required to construct the projections of the shortest distance between them, that is to say, of the straight

line which is perpendicular to each of them at the same time, and to find the length of this distance.

SOLUTION.—Conceive a plane parallel to the second of the two given straight lines to be passing through the first; which is always possible; for, if through any point in the first there be drawn a straight line parallel to the second, and this third straight line be supposed to move along the first line parallel to itself, it will generate the plane in question. Imagine also a cylindrical surface with a circular base having the second given straight line for its axis, and the required distance for its radius, and this surface will be touched by the plane in a straight line which will be parallel to the axis, and will cut the first straight line in one point. If through this point we draw a perpendicular to the plane, it will be the straight line required; for it will, in fact, pass through a point in the first straight line, and it will be perpendicular to it, because it is perpendicular to a plane passing through it; and it will cut the second straight line at right angles, because it is a radius of the cylinder whose axis is the second straight line.

It remains, then, but to construct successively the different parts of this solution.

1. To construct the traces of the plane drawn through the first straight line parallel to the second, find first the horizontal trace, A, of the first straight line (18), which will be a point in the horizontal trace of the plane; through the vertical trace of the first straight line conceive a straight line to pass parallel to the second; construct the projections of this parallel by drawing B E parallel to C D, and *b e* parallel to *c d;* and the horizontal trace E, of this parallel, will be a second point in the horizontal trace of the plane. If then the straight line A E be drawn, and produced till it cuts the axis L M in the point F, this will be the horizontal trace of the plane; and it is evident that, if through the points F and *b* the straight line F *b* be drawn, this will be the vertical trace of the plane.

2. To construct the line of contact of the plane with the

cylindrical surface, from any point whatever of the second straight line which is the axis of the cylinder, for instance, from the point C where it meets the horizontal plane, a perpendicular to the tangent plane must be drawn; and the intersection of this perpendicular with the plane will be a point in the line of contact. To find this intersection according to the method already described (14), construct first the projections of the perpendicular by drawing through the point C the straight line H G perpendicular to the trace A E, and through the point *c* the straight line *c* K perpendicular to the trace F *b*; then, after having produced H G until it meets A E in a point G, and L M in a point H, project the point G to *g*, and the point H to *h* upon the trace F *b*; draw the straight line *g h*, which, by its intersection with *c* K, will determine the vertical projection, *i*, of the point of intersection of the perpendicular with the tangent plane; and the horizontal projection, I, of the same point will be obtained by drawing *i* I perpendicular to L M, and producing it to meet H G in I. The projections, *i* and I, of the foot of the perpendicular being thus found, if I N be drawn parallel to C D, and *i n* parallel to *c d*, the projections of the line of contact of the plane with the cylindrical surface will be obtained; and the points N and *n*, where these projections meet those of the first given straight line, will be the projections of the point in this line, through which passes the required common perpendicular to the two given straight lines.

3. Knowing the projections N, *n*, of one of the points of the required perpendicular, to obtain the projections of this perpendicular we have only to draw through the points N and *n* the straight lines N P and *n p* perpendicular to the traces A E and F *b* respectively, and the parts N P and *n p* of these perpendiculars, comprised between the projections of the two given straight lines, will be the projections of the shortest distance required.

4. Lastly, to find the length of this shortest distance, construct it by the method explained in Art. 7.

The solution of the preceding question does not necessarily involve the consideration of a cylindrical surface touched by a plane. After having imagined a plane parallel to the two given straight lines, the intersection of two other planes, passing respectively through each of the given straight lines, and perpendicular to the first plane, will be the shortest distance required.

THE CONDITIONS WHICH DETERMINE THE POSITION OF A TANGENT PLANE TO ANY CURVED SURFACE: A REMARK RESPECTING DEVELOPABLE SURFACES.

32. In the various questions we have as yet resolved respecting tangent planes to curved surfaces, we have always supposed the point through which the tangent plane was required to be drawn, to have been situated in the surface, and to have been itself the point of contact; and this single condition was sufficient to determine the position of the plane. But it is not so, when the point through which the plane is required to pass is taken without the surface.

In order that the position of a plane may be determinate it must satisfy three different conditions, each of them equivalent to that of passing through a given point; while, in general, the property of touching a given curved surface, when the point of contact is not indicated, is equivalent to but one of these conditions. If then we propose to determine the position of a plane by conditions of this nature, there must in general be three such conditions. In fact, supposing we have three given curved surfaces, and that one of them is touched by a plane at any point, we may imagine this plane to move about the surface without ceasing to touch it; it can so move in all directions; remarking, however, that the point of contact will move upon the surface accordingly as the tangent plane changes its position, and that the direction of the motion of the point of contact will be the same as that of the motion of the plane. Supposing this motion to continue in a certain direction until the plane meets the second surface, and touches it in a certain point, the plane will then be a

tangent at the same time to the two first surfaces, and still its position will not yet be fixed. In fact, we may now suppose this plane to move about the two surfaces, always remaining in contact with both. It will not, as before, be free to move in all directions; but can move in one only. Accordingly as the plane changes its position, so the two points of contact will move on the surfaces to which they respectively belong; in such a manner, that if we suppose a straight line drawn through these two points, they will both move in the same direction with respect to this straight line, when the plane touches the two surfaces on the same side; and they will move in contrary directions, when the plane touches the surfaces, one on one side, and the other on the other. Finally, suppose this motion, which is the only one that can now take place, to continue until the plane touches the third surface at a certain point; then the position of the plane will be fixed, and can move no more without ceasing to be a tangent to at least one of the three surfaces.

We perceive then that, in order to determine the position of a plane by means of its indeterminate contacts with given curved surfaces, there must in general be three such surfaces. Thus, if it be proposed to draw a tangent plane to a given curved surface, this condition will only be equivalent to one of the three which the plane must satisfy: we are at liberty still to take two other conditions at pleasure, for instance, to make the plane pass through two given points, or what comes to the same thing, through a given straight line. If the plane is required to be a common tangent to two given surfaces, two conditions will be fixed; there will remain but one disposable, and the plane can be made in addition to pass through but one given point. Lastly, if the plane touches three given surfaces at the same time, there remains no disposable condition, and its position will be completely determined.

The preceding observations apply to curved surfaces in general; but it is necessary to make an exception with respect to all cylindrical, conical, and developable surfaces; for the con-

tact of a plane with a surface of this sort is not reduced to a single point; but extends the whole length of an indefinite straight line, which coincides with one of the positions of the generating line. The property of a plane being a tangent to but one of these surfaces is equivalent to two conditions, because it must in consequence pass through a straight line, and there will remain only one condition disposable, as, for instance, that it may pass through a given point. We could not then draw a plane to touch at the same time two of these surfaces, still less to touch three, unless there should be some particular circumstances which could render these conditions compatible with each other.

ON TANGENT PLANES TO SURFACES, DRAWN THROUGH GIVEN POINTS IN SPACE.

33. It may perhaps be useful, before proceeding farther, to give some examples showing the necessity there may sometimes be to draw tangent planes to curved surfaces from points taken without them. We will take the first of these examples from the construction of fortifications.

In stating the general principles of fortifications, we first suppose that the ground which surrounds the fortified place within cannon range, is quite level, and does not offer any eminence which can be of advantage to the besiegers. We, then, on this supposition determine the trace of the body of the work, the half-moons, the covered ways, and the out works; and we point out the command which different parts of the fortification ought to have over other parts, so that they may all contribute with the greatest effect towards their mutual defence. In order, afterwards, to apply these principles to the case in which the ground surrounding the place offers some elevation of which the besiegers may take advantage, and from which the fortification must be defiladed, only one new consideration has to be introduced. If there be but one such elevation, we select two points in the place, through which we suppose a plane to pass touching the height from which we wish

to be defiladed: this plane is called the plane of defilement, and the same relief is to be given to each part of the fortification above the plane of defilement, as it would have had above the horizontal plane, supposing the ground had been quite level. By this proceeding the different parts of the work will have the same command over each other, and over the neighbouring height, as on horizontal ground; and the fortification has the same advantages as in the first case. In selecting the two points through which the plane of defilement is to pass, the two following conditions should be observed: 1st, that the inclination of the plane to the horizon be as small as possible; so that, the terre-pleins having less slope, the service of the defence may be rendered less difficult: 2ndly, that the relief of the fortification above the natural level of the ground be as small as possible, in order to entail less labour and expense in its construction.

If there happen to be two heights in the surrounding country from both of which the fortification is to be defiladed, the plane of defilement must be at the same time a tangent to each of the surfaces of these two eminences; and there remains in addition but one disposable condition to fix its position; namely, to select a point in the fortification through which this plane is to pass, so as best to satisfy the conditions enunciated in the former case.

34. The second example which we shall give will be again taken from the art of painting.

The surfaces of bodies, especially when polished, are covered with shining points, the brightness of which depends upon the degree of light cast on them from luminous bodies. The more polished the surfaces, the greater is the brilliancy of these points, and the smaller their extent. When the surfaces are dull, the shining points have much less brightness, and occupy much more space.

The following conditions determine the shining point for each surface: namely, that the incident ray of light, and the reflected ray, which reaches the eye of the spectator, lie in one

plane perpendicular to the tangent plane at this point, and make equal angles with this plane, since the shining point performs the part of a mirror, and reflects to the eye a part of the image of the luminous object. The determination of this point requires the greatest precision; and even though the design should be most perfectly correct, and the apparent outlines be traced with a mathematical precision, the commission of the slightest error in the position of the shining point would produce very considerable faults in the appearance of the forms represented. Of this the following is a very striking instance. The surface of the ball of the eye is polished; it is moreover coated with a slight film of moisture, which still further increases the polish; and consequently when one notices an open eye, he perceives upon its surface, a brilliant point of great lustre, very small in size, the position of which depends upon those of the illumining body, and of the observer. If the surface of the eye were perfectly spherical, the eye could turn about its vertical axis without the least alteration in the position of the brilliant point; but this surface being somewhat lengthened in the direction of the axis of vision, the brilliant point changes its position, as the eye turns about its vertical axis. Long experience having rendered us very sensible to this change, it forms a considerable item in our judgment respecting the direction of the ball of the eye. It is chiefly by the difference of the positions of the brilliant points upon the balls of the two eyes of a person that we judge whether he squints or no, that we perceive when he is looking at us, and, when he is no longer looking at us, observe on which side he directs his view. In giving this example it must not be supposed that in a picture we consider it essential to determine geometrically the brilliant point upon the ball of the eye; we have merely taken this as an instance of how considerable errors may arise in the apparent form of an object from slight mistakes in the position of this point; although the outline remains the same.

ON THE TANGENT PLANE TO THE SURFACE OF ONE OR OF SEVERAL SPHERES.—REMARKABLE PROPERTIES OF THE CIRCLE, SPHERE, CONIC SECTIONS, AND CURVED SURFACES OF THE SECOND DEGREE (pl. III. IV. and V., figs. 16 to 22).

35. Of all surfaces, that of the sphere is one of the most simple; it may be regarded as being generated in a similar manner with a great number of different surfaces: we could, for instance, rank it among surfaces of revolution, without making any remark about it in particular. But the regularity of its figure gives rise to some remarkable results, some of which are interesting for their novelty, and which we proceed to investigate, not so much for their own sakes, as for the purpose of acquiring that facility in the consideration of three dimensions, which will be needed for objects of a more general nature, and of a more useful character.

36. PROB. 14.—Through a given straight line to draw a plane which shall touch a given sphere.

SOLUTION.—*First method.*—Let A and *a* (pl. IV., fig. 16) be the two projections of the center of the sphere; B C D the projection of the great horizontal circle; E F and *e f* the two indefinite projections of the given straight line. Suppose a plane to be drawn through the center of the sphere perpendicular to the given straight line, intersecting it in a point whose projections G and *g* are found (14).

This being premised, it is evident that through the given straight line two tangent planes can be drawn to the sphere, of which the first will touch it on one side, the second will touch it on the other, and between which it will be placed; this will enable us to determine two different points of contact, whose projections we have to construct.

For this purpose suppose perpendiculars to be let fall from the center of the sphere upon each of the tangent planes; they will each pass through the point where the corresponding plane

touches the sphere; and they will both be situated in the plane which is perpendicular to the given straight line; the two points of contact will lie, then, in the circumference of the great circle, which is the intersection of the surface of the sphere with the last mentioned plane; and the lines of intersection which this plane makes with the two tangent planes, will be tangents to the circumference of this circle.

Again, in the perpendicular plane, and through the center of the sphere, imagine a horizontal line to be drawn, of which the vertical projection is found by drawing the horizontal line $a\,h$, and the horizontal projection by letting fall the perpendicular A H from A upon F E; and, conceiving the perpendicular plane to turn about this horizontal line, as upon a hinge, until it comes into a horizontal position, it is evident that its intersection with the surface of the sphere will then come into coincidence with the circumference B C D, that the two points of contact will lie in this circumference, and that if we construct the point J, at which the intersection of the perpendicular plane with the given line will arrive by this movement, then the tangents J C, J D, drawn to the circle B C D will determine the two points of contact in the position which they will then occupy. Now it is easy to construct the point J, or, what amounts to the same thing, to find its distance from the point H: for the horizontal projection of this distance is G H, and the difference of the vertical heights of its extremities is $g\,g'$; if, then, we set off G H upon the horizontal line $a\,h$, from g' to h, the hypotheneuse $h\,g$ will be the length of this distance; and by measuring off $g\,h$ upon E F from H to J, and drawing the tangents J C, J, D, the two points of contact C, D, will be determined in the position which they assume, when the perpendicular plane is turned down upon the horizontal plane.

Now in order to find the projections of the points of contact for the positions which they really have, we must suppose the perpendicular plane to return into its original position by again turning about the horizontal line A H, as upon a hinge, and

carrying with it the point J, the two tangents J C, J D, produced till they cut A H in the points K, K′, and the chord C D, which will also cut the same straight line A H at a point N. It is evident that, during this motion, the points K, K′ and N, which are upon the axis of rotation will remain fixed, and that the two points of contact C, D, will describe circular arcs which will lie in planes perpendicular to this axis, and whose horizontal projections will be obtained by letting fall from the points C, D, upon A H, the perpendiculars C P, D Q, produced indefinitely. The horizontal projections, then, of the two points of contact will be found in the straight lines C P, D Q. But during the retrograde movement of the perpendicular plane, the two tangents J C K′, J K D, will always pass through the respective points of contact; and when this plane has come into its original position, the point J will be again projected on G, and the two tangents will be projected along the straight lines G K′, G K; and consequently, each of these two last lines must contain the horizontal projection of one of the points of contact. The intersections, therefore, of these two straight lines with C P, and D Q, respectively, will determine the horizontal projections R and S of the two points of contact, the line joining which, since it is the projection of the chord C N D, will pass through the point N.

To find the vertical projections of these points, draw first the indefinite lines R *r*, S *s*, perpendicular to L M. Then project the points K, K′, to *k*, *k*′; and, the straight lines *g k*, *g k*′, drawn through the point *g*, will be the vertical projections of the two tangents. These straight lines will, therefore, contain the projections of the respective points of contact; and the points *r*, *s*, in which they intersect the verticals, R *r*, and S *s*, will be the projections required.

Having determined the horizontal and vertical projections of the two points of contact, in order to construct the horizontal traces of the two tangent planes, imagine a straight line to pass through each of the points of contact parallel to the

given straight line. These straight lines will, then, be situated in the respective tangent planes, and their horizontal and vertical projections will be obtained by drawing R U, S V, parallel to E F, and *r u*, *s v*, paralallel to *ef*. Construct the horizontal traces U, V, and T of these two straight lines and of the given line, respectively, and the lines T U, T V will be the horizontal traces of the two tangent planes.

Instead of supposing two more straight lines to pass through the points of contact, we might have found the traces of the two tangents G R, G S, which would have answered the same purpose. The vertical traces of the tangent planes may be found by the method we have already frequently employed.

This solution might be rendered much more elegant, by making the two planes of projection pass through the very center of the sphere. In this case the two projections of the sphere would coincide with the same circle, and the figure would occupy a much smaller space. We have separated the two projections merely to give greater clearness to the explanation; and the student can easily give to the construction all the conciseness which it is capable of receiving.

37. *Second method.*—Let A and *a* (pl. IV. fig. 17) be the two projections of the center of the sphere, A B, or *a b*, its radius, B C D, the projection of its great horizontal circle, and E F, *ef*, the projections of the given straight line. If the plane of the great horizontal circle be supposed to be produced until it cuts the given straight line at a certain point, the vertical projection of this plane will be obtained by drawing through the point *a* horizontal straight line, *b a g*, of indefinite length; the point *g*, where this horizontal line cuts *ef*, will be the vertical projection of the point of intersection of this plane with the given straight line, and the horizontal projection, G, of this point will be obtained by projecting *g* on E F.

If, now, this same point be conceived to be the vertex of a conical surface which envelopes the sphere, and of which all the generating lines touch it each in one point, the projections of the

two horizontal generating lines of this conical surface will be obtained by drawing through the point G, the two straight lines G C, G D to touch the circle B C D in the points C, D, which will be easily determined. The conical surface then will touch that of the sphere in the circumference of a circle of which C D will be the diameter, the plane of which will be perpendicular to the axis of the cone, and consequently vertical, and whose horizontal projection will be the straight line C D.

Conceiving further two tangent planes to the conical surface to pass through the given straight line, each of them will touch the surface along the entire length of one of its generating lines, which will thus be situated at the same time both in the conical surface, and in the tangent plane; and, since this generating straight line also touches the surface of the sphere in a point which lies in the circumference of the circle projected on C D, it follows that this point is at the same time in the conical surface, in the plane which touches this surface, in the surface of the sphere, and in the circumference of the circle projected on C D, and that it is a common point of contact to all of them. Consequently, in the first place, the two tangent planes to the conical surface are also tangents to the surface of the sphere, and are those the position of which we are seeking to determine; in the second place, their points of contact with the sphere, being in the circumference of the circle projected on C D, will themselves be projected somewhere on this straight line; and thirdly, the straight line which passes through the two points of contact, being contained in the plane of the same circle, will be itself projected on some part of C D.

Again, repeating the same process with the plane of the great circle parallel to the vertical plane of projection, as we have just gone through with the plane of the great horizontal circle, the horizontal projection of that plane will be the straight line B A H of indefinite length, parallel to L M; the point in which it meets the given straight line will have its horizontal projection at the intersection, H, of the two straight lines E F,

B A H; and its vertical projection, *h*, is found by projecting H upon *ef*. Conceiving then another conical surface, having this point for its vertex, to envelope the sphere in the same manner as before, the vertical projections of the two extreme generating straight lines of this surface will be obtained, by drawing through the point *h* the straight lines *h* K, *h* I, to touch the circle *b* K I in the points K, I. This second conical surface will touch that of the sphere in the circumference of another circle, of which K I will be the diameter, and which, from its plane being perpendicular to the vertical plane of projection, will also have K I for its vertical projection. The circumference of this circle also will pass through the two points in which the required tangent planes touch the sphere; the vertical projections, therefore, of these two points will lie somewhere in K I, and, therefore, also the vertical projection of the straight line, which joins them will be in some part of K I.

Thus it will be perceived that C D is the horizontal, and K I the vertical projection of the straight line which joins the points of contact; and that this chord of contact meets the plane of the great horizontal circle at a point, whose vertical projection is the intersection, *n*, of K I with *b a g*, and whose horizontal projection, N, is found by projecting the point *n* on C D.

This being effected, conceive the plane of the vertical circle, which is projected on C D, to revolve about its diameter as upon a hinge, so as to take up a horizontal position, and to carry with it during this motion the two points of contact through which the circumference passes, as well as the chord which joins them. The construction of this circle in its new position will be obtained by describing the circle C P D Q upon C D as a diameter; and, if the position which is taken up by the chord of contact were constructed, it would cut the circumference C P D Q in two points, which would determine the points of contact, as they would appear on this circumference in its horizontal position.

Now in this movement, the point N of the chord of contact, lying on the axis of motion, remains fixed. This chord, there-

fore, when it has become horizontal, must still pass through that point. Moreover, the point in which it meets the plane of the great circle parallel to the vertical plane of projection, a point, the horizontal projection of which is at the intersection, O, of the two straight lines C D, B A H, and of which the vertical projection, *t*, is obtained by projecting the point O on K I; this point, in its movement about the axis C D, describes a fourth part of a vertical circle perpendicular to C D, the radius of which is the vertical *o t*; so that, drawing through the point O a perpendicular to C D, and on this perpendicular setting off *o t* from O to T, the point T will be one of the points of the chord of contact, when it has become horizontal. If, therefore, a straight line be drawn through the points N and T, the two points P, Q, in which it meets the circumference C P D Q, will be the positions of the two points of contact, when the vertical plane has been turned down into the horizontal position.

In order to obtain the horizontal projections of the same two points in their natural positions, the circle C P D Q must be conceived to return to its original position by revolving about the same axis C D. In this movement, the two points P, Q, will describe quadrants of circles in vertical planes, perpendicular to C D, and whose horizontal projections will be the perpendiculars, P R and Q S, let fall upon C D. The horizontal projections, therefore, of the two points of contact will lie upon the straight lines P R and Q S, respectively; and it has been shown that they must also be situated on C D; consequently they will be at the two points, R and S, in which P R and Q S meet C D.

The vertical projections, *r* and *s*, of the same two points will be obtained by projecting R and S upon K I, or, what amounts to the same thing, by measuring off from the horizontal line *b a g* upon the vertical lines R *r*, S *s*, the two distances *r r'* and and *s s'* equal to P R and Q S, respectively.

The horizontal and vertical projections of the two points of contact having been constructed, the traces of the two tangent planes are to be determined as in the first solution.

This second solution may also be rendered much more concise, by making the planes of projection pass through the center of the sphere; a construction which reduces its two projections to one and the same figure.

38 The considerations upon which this last solution of the problem has been founded, will lead us to the discovery of some remarkable properties of the circle, sphere, conic sections, and curved surfaces of the second degree.

We have just seen that the two conical surfaces circumscribed about the sphere will each touch it in the circumference of a circle, and that these circumferences will both pass through the two points in which the sphere is touched by the tangent planes. This property is not confined to the two conical surfaces which we have been considering; it applies equally to all those which have their vertices in the given straight line, and which are circumscribed in the same way about the sphere. If, then, we conceive a primary conical surface having its vertex upon the given straight line, to be circumscribed about the sphere, and if this surface be imagined to move in such a manner that, while its vertex runs along the straight line, it never ceases to be circumscribed about and touch the sphere, in every different position, it will touch the sphere in the circumference of a different circle: all these circumferences will pass through the same two points, which will be the points in which the sphere is touched by the two tangent planes; and the planes of these circles will all intersect in the same straight line, which will be the chord of contact. Conceiving, moreover, a plane to pass through the given straight line and the center of the sphere, this plane, which will pass through the axes of all the conical surfaces, will be perpendicular to the planes of all the circles of contact, and, consequently, to the straight line which is their common intersection, and it will intersect all these planes in straight lines passing through the same point.

Conversely, a sphere and a straight line being given, if any number of planes be conceived to pass through the straight line,

the intersection of each of which with the sphere will form a circle, and if a series of right conical surfaces, having these circles for their respective bases, be conceived to circumscribe the sphere, then the vertices of all these conical surfaces will lie in one other straight line.

39. By merely considering what takes place in the plane drawn through the given straight line and the center of the sphere, we are led to the two following propositions, which are direct corollaries of the preceding article.

"A circle whose center is A (pl. III., fig. 18 and 19), and any straight line B C, in the same plane being given; if, after having drawn two tangents to the circle from any point D in the straight line, and the straight line, E F, which passes through the two points of contact, the point D be conceived to move along the straight line, and to carry with it the two tangents, without their ever ceasing to touch the circle, the two points of contact and the chord of contact E F will continually change their position; but this chord will always pass through one and the same point N, which lies in the perpendicular A G let fall, from the center of the circle upon the straight line."

"Conversely, if through a point N taken in the plane of the circle, any number of straight lines, as E F, be drawn, each of which cuts the circumference of the circle in two points, and if through these two points, the two lines E D, F D, be drawn touching the circle, and intersecting each other somewhere in D, the locus of all the points of intersection found in the same manner will be a straight line B C perpendicular to A N."

The property which we have just stated does not belong to the circumference of a circle in consequence of all its points being equally distant from its center, but because it is a curve of the second degree; and the same property applies to all the conic sections.

In fact, let A E B F (pl. III. fig. 20) be any conic section whatever, and C D any given straight line in its plane: imagine the curve to revolve about one of its axes, A B, and generate a

surface of revolution, and suppose two tangent planes to this surface to be drawn through the straight line C D; the two planes will have each their own point of contact. This being premised, if taking any point H of the straight line C D for the vertex, a conical surface be conceived to be circumscribed about, and touch the surface of revolution, it will touch this last surface in a curve which will necessarily pass through the two points of contact with the tangent planes. This curve will be a plane curve; its plane, which will be perpendicular to that of the given conic section, will be projected on this last along a straight line E F, and this straight line will pass through the points of contact of the tangents to the conic section, drawn through the point H. If now the conical surface be supposed to continue to be circumscribed about, and to touch the surface of revolution, while its vertex H moves along the straight line C D; in each of its positions the curve of contact will have the same properties of passing through the two points of contact with the tangent planes, of being a plane curve, and of having its plane perpendicular to the conic section.

The planes of all the curves of contact, then, will pass through the chord of contact, which is itself perpendicular to the plane of the conic section; and therefore, the projections of all these planes will be straight lines which will all pass through N, the projection of the chord of contact.

40. Moreover, this proposition is itself only a particular case of another still more general, which holds good in three dimensions, and which we proceed to enunciate.

"Having given in space any curved surface whatsoever of the second degree, and a conical surface which circumscribes and touches it, the vertex of which, may be at any point whatsoever, if the conical surface moves, so that, while its vertex passes along any straight line whatever, it still continues to be circumscribed about, and touch the first surface, the plane of the curve of contact of the two surfaces will always pass through the same straight line, which will be determined by the points of

contact of the surface of the second degree with the two tangent planes which pass through the straight line containing the vertices; and if the conical surface moves, so that its vertex may always be in the same plane, the plane of the curve of contact will always pass through the same point.

41. PROB. 15.—To draw a plane through a given point which shall touch the surfaces of two given spheres.

SOLUTION.—Let A, *a*, (pl. V. fig. 21) be the two projections of the center of the first sphere; B, *b*, those of the center of the second; and C, *c*, those of the given point. After having drawn A B, *a b*, the projections of the straight line which passes through the two centers, and having constructed G E F, *g e f*, H I K, *h i k*, the projections of the great circles of the two spheres parallel to the planes of projection, imagine a conical surface circumscribing and touching both surfaces at the same time. The vertex of this surface will lie in the straight line which passes through the two centers; and drawing the two common tangents, E H, F K, to the two circles G E F, H I K, these tangents will intersect each other at a point D in the straight line A B, which point will be the horizontal projection of the vertex of the cone: the vertical projection of the same point will be obtained by projecting the point D to *d* on *a b* produced. Lastly, draw C D, *c d*, the projections of the straight line joining the vertex of the cone, and the given point. If, now, two tangent planes to the conical surface be conceived to pass through this last line, they will touch it each in a line of contact coinciding with one of its generating lines, and consequently they will be both tangents to the two spheres. The question is, therefore, reduced to that of drawing two tangent planes to the surface of one of the spheres through the straight line joining the vertex of the cone and the given point, which can be effected in the same manner as in the preceding problem, and the two planes will likewise touch the second sphere.

It is to be remarked that two conical surfaces may be circumscribed at the same time about the two spheres, and the

manner in which they are respectively situated may be perceived by drawing two pair of tangents to the circles E F G, H I K. The two exterior tangents, D E, D F, intersecting in D, are the horizontal projections of the first pair, and the two interior tangents, intersecting in a point, D′, upon the line A B joining the centers, are the horizontal projections of the second pair. D′ then is the horizontal projection of the vertex of the second conical surface: and its vertical projection is obtained by projecting D′ on $a\,b$ at the point d'. The two tangent planes to this conical surface will also each touch the two spheres; but they touch the first on one side, and the second on the other. Thus four different planes satisfy the problem; for two of which, the two spheres are on the same side of the plane; and for the other two, they are on contrary sides.

42. Prob. 16—To draw a common tangent plane to three spheres which are given in magnitude and position.

Solution.—Conceive a common tangent plane to the three spheres, and then imagine a conical surface circumscribed about the first two of them and touching them both, the tangent plane will touch this conical surface in a straight line coinciding with one of its generating lines, and will pass through the vertex of the cone. Again imagining a second conical surface, circumscribed about the first and third spheres, the same tangent plane will touch this also along one of its generating lines, and will therefore pass through its vertex. Imagining, lastly, a third conical surface, which circumscribes and touches the second and third spheres, the tangent plane again will touch this also along one of its generating lines, and pass through its vertex. Thus the vertices of the three surfaces will be in the tangent plane; but they will also be in the plane which passes through the centers of the spheres, and which contains the three axis: therefore, being at the same time in two different planes, they will be situated in one straight line. It follows from this, that if the horizontal and vertical projections of two of these vertices be constructed, as in the preceding problem, the lines joining these projections will also pass through the projections of the third

vertex, and will be the projections of a straight line situated in the tangent plane. The question reduces itself then, to that of drawing through a given straight line a tangent plane to any one we please of the three spheres, which may be effected by the preceding methods; and this plane will also be a tangent to the two others.

43. It is to be remarked that, since one can always conceive two different conical surfaces to be circumscribed about any two spheres, and to touch them both, the first having its vertex in the production of the line joining their centers, and the second in the straight line joining them, in the preceding question, there will be six conical surfaces, of which three will be circumscribed exteriorly to the three spheres taken two and two together, and the other three will have their vertices between the spheres. The vertices of these six cones will be distributed three by three on four straight lines, through each of which may be drawn two planes touching at the same time the three spheres. Consequently, eight different planes satisfy this problem: two of them touch the three spheres on the same side; the six others are so placed that they touch two of the spheres on one side, and the third on the other side.

44. These considerations lead to the following proposition—

"Any three circles in one plane being given in magnitude and position; (pl. V., fig. 22), if, considering them two and two together, the exterior tangents be drawn, and produced to meet, the three points of intersection D, E, F, thus obtained, will be in one and the same straight line."

"For if we imagine the three spheres of which these circles are the great circles, and a plane which touches them all three externally, this plane will also touch the three conical surfaces circumscribed about the spheres taken two and two together, and will pass through their three vertices D, E, F. But these three vertices are also in the plane of the three centers; they are therefore upon two different planes, and consequently in one straight line." "If considering the same circles two by two the interior

tangents which cross each other be drawn, the three new points of interection G, H, I, will be two by two in a straight line with one of the first three, so that the six points D, E, F, G, H, I, will be the intersections of four straight lines."

Moreover, this proposition is but a particular case of the following, which holds good in three dimensions.

Any four spheres being given in magnitude and position in space, if, considering these spheres two by two, six exterior conical surfaces be conceived to be circumscribed about them, the vertices of the six cones will be in the same plane and in the intersections of four straight lines; and if six other conical surfaces be conceived to be circumscribed interiorly, that is to say, so as to have their vertices between the centers of two spheres, the vertices of these six new cones will be, three by three together, in the same plane with three of the former.

ON THE TANGENT PLANE TO A CYLINDRICAL SURFACE, A CONICAL SURFACE, AND TO A SURFACE OF REVOLUTION, DRAWN THROUGH POINTS WITHOUT THE SURFACES (pl. V. and VI., figs. 23 to 25).

45. Prob. 17.—Through any point taken arbitrarily to draw a tangent plane to a given cylindrical surface.

Solution.—Let E I F K (pl. V., fig. 23) be the trace of the cylindrical surface upon the horizontal plane, which trace we suppose given. Let A B, *a b*, be the two given projections of the straight line to which the generating straight line must always be parallel; and C, *c*, those of the given point. If through this point a straight line be conceived to pass parallel to the generating line, this straight line will lie in the required tangent plane, and the points in which it cuts the planes of projection will lie on the traces of the tangent plane. Drawing then through the point C, C D, parallel to A B, and through the point *c*, *c d*, parallel to *a b*, the two projections of this straight line will be obtained, and if, after having produced *c d* until it meets L M

at a point *d*, the point *d* be projected upon C D at D, the point D will be the horizontal trace of this straight line, and, consequently, a point in the horizontal trace of the tangent plane. Now this trace of the tangent plane must be a tangent to the curve E I F K; therefore, if through the point D there be drawn to this curve all possible tangents such as, D E, D F, &c., we shall have the horizontal traces of all the tangent planes, which can pass through the given point. Again, if through the points of contact E, F, &c., the straight lines E G, F H, &c., of indefinite length be drawn parallel to A B, these will be the horizontal projections of the generating straight lines, in which the different tangent planes touch the cylindrical surface; and, lastly the vertical projections *e g*, *f h*, &c., of these generating straight lines, or of these lines of contact, will be found by projecting the points E, F, &c. to *e*, *f*, &c., upon the vertical plane, and by drawing through these last points, straight lines of indefinite length parallel to *a b*. With respect to the vertical traces of the tangent planes, they will be found by the method used in fig. 12, and described in art. 28.

46. Prob. 18.—Through any point taken arbitrarily, to draw a tangent plane to a given conical surface.

Since the solution of this question differs very little from that of the preceding, we shall merely indicate the construction by fig. 24, pl. V., in which the curve E F G H is the given trace of the conical surface, A and *a* are the given projections of the vertex, and C, *c* are those of the given point through which the tangent is to pass.

47. Prob. 19.—Through a given straight line, to draw a tangent plane to a given surface of revolution.

Solution.—We will suppose the axis of the surface of revolution to be perpendicular to one of the two planes of projection, which will not affect the generality of the solution, since we are always at liberty to give these planes such a position, that this condition may be satisfied.

Let, then, A (pl. VI. fig. 25) be the given horizontal projection

of the axis of the surface, *a a'* its vertical projection, *a p i a'*, the generating curve of the surface, and B C, *b c*, the two given projections of the straight line, through which the tangent plane is to pass. From the point A let fall upon B C the perpendicular A D, which will be the horizontal projection of the shortest distance between the axis and the given straight line, and let D be projected upon *b c* at the point *d*.

This being done, conceive first the tangent plane to be drawn; then imagine the given straight line to revolve about the axis of revolution, always keeping at the same distance from this axis, and retaining the same inclination to the horizontal plane, and to carry along with it the tangent plane, so that this always touches the surface: it is evident that by virtue of this movement the point of contact of the surface and plane will change its position; but since the tangent plane always retains the same inclination, this point of contact will always be at the same height on the surface, and will move in the circumference of a horizontal circle, the center of which will be in the axis. Moreover, the given straight line will by its motion generate a second surface of revolution about the same axis, to which again the tangent plane will be a tangent in all its positions. In fact, conceiving a plane to pass through the axis, and through the point of contact of the tangent plane with the first surface, that plane will intersect the generating straight line at a point which will be the point of contact of the same tangent plane with the second surface; for, independently of the generating straight line through which this tangent plane passes at this point, it passes also through the tangent of the horizontal circle at the same point, since it also passes through the tangent of the horizontal circle at the point of contact with the first surface, and by the property of surfaces of revolution, these two tangents are parallel.

Since this question is to be resolved by means of the second surface of revolution, it is necessary to construct the curve in which this surface is cut by a plane drawn through the axis;

and we will suppose that this plane is parallel to the vertical plane of projection, and has consequently for its horizontal projection a straight line A F parallel to L M.

In the given straight line let any point whatever be taken whose projections are E and *e*, and let us find the point in which in the course of its motion it meets the plane of section. In the first place, this point will describe an arc of a horizontal circle about the axis, whose horizontal projection is obtained by describing with center A and radius A E, an arc E F meeting the straight line A F in some point F, and its vertical projection, by drawing through the point *e* the horizontal straight line *ef* of indefinite length. The point F will therefore be the horizontal projection of the meeting of the describing point with the plane of section; and, if the point F be projected upon *ef* at *f*, the point *f* will be the vertical projection of this meeting, and consequently a point of the section. If the same operations be gone through for as many other points on the straight line as we please, so many points *g*, *f*, *r*, *n*, &c., will be obtained, through which the required curve is to be drawn.

This being done, suppose the given straight line, and the tangent plane by their simultaneous rotation about the axis, to have come into such a position, that the tangent plane is perpendicular to the vertical plane of projection. In this position its projection on this plane will be a straight line, and this straight line will touch the two curves *a p i a'*, *g r n f* at the same time. If, then, all the common tangents to these two curves be drawn, such as *g i*, *n p*, the projections of all the tangent planes which satisfy the question will be obtained, these planes holding the positions they have taken, when by the rotation, they have become successively perpendicular to the vertical plane. The points of contact, *i*, *p*, of these tangents, with the generating line of the first surface, will determine the heights of the points of contact of this surface with all the tangent planes, respectively; consequently, if through these points the indefinite horizontal lines *it*, *ps*, be drawn, they will contain the vertical projections of the points of contact of

the surface with the planes; and if with the point A as a centre, and with radii equal to *i t* and to *p s*, respectively, the circular arcs I K, P Q, be described, these arcs will contain the horizontal projections of the same points. There remains then nothing more, in order to completely determine them, but to find upon what meridians of the surface of revolution they lie: and this is to be accomplished by the aid of the points of contact *g*, *n*.

In order to effect it, after having projected the points *g* and *n*, upon A G at G and N, if with the point A as centre, and with distances successively equal to A G and A N, the circular arcs G H, N O, be described, and produced until they intersect the straight line B G in the points H and O, these arcs express the respective quantities of rotation that the straight lines, which respectively pass through the points of contact of each tangent plane with the two surfaces, have been obliged to make in order to come into the vertical plane parallel to the plane of projection. Consequently, the horizontal projections of the same straight lines, considered in their natural positions, will be obtained by drawing the straight lines A H, A O, through the point A; and therefore, lastly, the points K, Q, where these last straight lines cut the corresponding arcs, I K, P, Q, will be the horizontal projections of the points of contact of the first surface with the tangent planes drawn through the given straight line.

The vertical projections of the same points will be obtained by projecting the points K, Q, to *k*, *q*, upon the respective horizontal lines *i t*, *p s*.

Having determined the horizontal and vertical projections of the points of contact, the traces of all the tangent planes are to be constructed by the methods already employed.

This method may easily be generalized, and applied to surfaces generated by any curves whatever, constant in form, and variable in position in space.

CHAPTER III.

ON THE INTERSECTIONS OF CURVED SURFACES.

DEFINITION OF CURVES OF DOUBLE CURVATURE.

48. When the generations of two curved surfaces are completely determinate and known; when for neither of them is anything left arbitrary respecting the succession of all the points in space through which they pass; when for each of these points, one of the two projections being taken at pleasure, the other projection can always be constructed; if the two surfaces have any common points in space, the position of all such common points is absolutely determinate; it depends both upon the form of the two curved surfaces, and upon their respective positions, and is of such a nature that it can always be deduced from the definition of the generations of the surfaces, of which it is a necessary consequence.

The succession of all the points common to two determinate curved surfaces, forms generally a certain curved line in space, which in some particular cases lies in one certain plane, and has but one curvature; which in cases of infinitely rare occurrence becomes a straight line and has no curvature, and which in cases infinitely rarer still is reduced to one single point; but which, in the general case is what is called a curve of double curvature, because it usually participates in the curvatures of the two surfaces, on each of which it is situated at the same time, and of which it is the common intersection.

THE GENERAL METHOD FOR DETERMINING THE PROJECTIONS OF THE INTERSECTIONS OF CURVED SURFACES. MODIFICATION OF THIS METHOD IN SOME PARTICULAR CASES. (Pl. VI., fig. 26).

51. In order to impart greater clearness to the explanation of this method, we shall not present it at once with all the elegance

of which it is susceptible; but arrive at this point by degrees. Moreover, the enunciation will be general and applicable to any two surfaces whatever; and although the letters employed will refer to pl. VI., fig. 26, which presents the particular case of two conical surfaces, with circular bases, and vertical axes, it is nevertheless to be conceived that the surfaces under consideration can either of them be of any other kind, as well as conical.

52. Prob. 20.—First General Prob.—The generations of two curved surfaces being known, and all the data which fix these generations being determinate on the planes of projection, to construct the projections of the curve of double curvature in which the two surfaces intersect each other.

Solution.—Conceive a succession of indefinite planes placed in space according to some manner agreed upon: for example, suppose, first, that all these planes are horizontal. In this case the vertical projection of each of them will be a horizontal straight line of indefinite length; and since the distances between them are quite arbitrary, suppose as many horizontal straight lines ee', ee', ee', &c. (pl. VI., fig. 26) as you please, to have been drawn in the vertical plane of projection, and this set of straight lines to be the vertical projection of the set of planes which we have imagined. Having arrived at this point, the same operation is to be gone through for each of these planes, and with respect to the straight line ee' which is its projection, which we proceed to point out for that particular one which is projected to E E′.

The plane E E′ cuts the first surface in a certain curve, which can always be constructed if the generation of the surface is known; for this curve is the succession of the points in which the plane E E′ is pierced by the generating line in all its different positions. Lying in a horizontal plane, it will have its horizontal projection equal, similar, and similarly situated to itself: this projection, therefore, can be constructed; suppose it, then, to be the curve F G H I K.

The same plane E E′ will also cut the second surface in

another plane horizontal curve, the horizontal projection of which can be constructed; suppose it, then, to be the curve F G O P N.

Now, the two curves in which the same plane E E′ cuts respectively the two surfaces may intersect each other, or they may not: if they do not intersect, however far produced, this will be a proof that at the height of the plane E E′ the two surfaces have no point in common; but if these two curves do intersect, they will do so in a certain number of points, which will be common to the two surfaces, and will therefore be so many points of the required intersection. In fact, inasmuch as the points of intersection of the two curves are in the first curve, they are in the first of the two surfaces; and inasmuch as they are in the second curve, they are also in the second surface; inasmuch, then, as they are in both the curves at once, they are also in both the surfaces.

Now, the horizontal projections of the points in which the two curves intersect, must be situated both on the horizontal projection of the first, and on that of the second; consequently F, G, &c., the points of intersection of the curves F G H I K and F G O P N, will be the horizontal projections of so many points in the required intersection of the two curved surfaces. To obtain the vertical projections of the same points, it is to be observed that these points are all comprised in the horizontal plane E E′, and that their vertical projections must be situated on the line E E′. If, therefore, the points F, G, &c., be projected to *f*, *g*, &c., upon E E′ the vertical projections of these points will be obtained.

If the same process be gone through for all the other horizontals *e e′*, *e e′*, &c., as for E E′, there will be found for each of them a succession of new points F, G, &c., in the horizontal plane of projection, and in the vertical plane of projection, a succession of new points *f*, *g*, &c. Then if a branch of a curve be made to pass through all the points F, &c., another branch through all the points G, &c., and so on, the assemblage of all these branches,

which may sometimes run one into the other, will be the horizontal projection of the intersection of the two surfaces; and in the same way, if a branch of a curve be made to pass through all the points f, &c., another branch through all the points g, &c., and so on, then, the assemblage of all these branches, which may sometimes run one into the other, will be the vertical projection of the required intersection.

53. The method which we have just explained is general, supposing a succession of horizontal planes to have been selected for the system of cutting planes. It is, however, to be observed that in certain cases, the choice of the system of cutting planes is not a matter of indifference, that such a choice may sometimes be made as to render more simple and elegant the resulting constructions, and that it may even perhaps be advantageous, instead of a system of planes, to employ a succession of curved surfaces, which differ from each other but in one of their dimensions.

To construct the intersection of two surfaces of revolution whose axes are vertical, the most advantageous system of planes is a succession of horizontal planes; for each of these planes cuts the two surfaces in the circumferences of circles, whose centers are in their respective axes, whose radii are equal to the ordinates of the generating curves, taken at the height of the cutting plane, and whose horizontal projections are circles known both in magnitude and position. In this case, then, all the points of the horizontal projection of the intersection of the two surfaces are found by the intersections of circular arcs. We perceive, further, that if the surfaces of revolution have their axes parallel to each other, but not vertical, the planes of projection should be changed, and so chosen, that one of them may be perpendicular to the axes.

54. If it were required to construct the intersection of two conical surfaces having any bases whatever, and whose traces on the horizontal plane were given, the system of horizontal planes would lead to operations which would be too tedious; for each of the horizontal planes would by its intersection with

the two surfaces form curves similar, indeed, to the traces of the respective surfaces, but not equal to them: they would have to be constructed, each of them separately, by points; whilst, if, after having drawn a straight line through the given vertices of the two cones, we employ a system of planes which pass through this straight line, each of these planes will cut the two conical surfaces in four straight lines; and these straight lines, which will lie in one plane, will intersect each other, independently of the vertices, in four points, which are situated in the intersection of the two surfaces. In this way, then, each point in the horizontal projection of this intersection, will be constructed by the intersection of two straight lines.

55. Neither would a system of horizontal planes be the most favorable for determining the intersection of two cylindrical surfaces, whose bases are of any sort whatever, and whose generating lines are inclined in different directions. Each of these planes would, indeed, intersect the two surfaces in curves similar and equal to their respective traces; but the curves, not corresponding vertically with the traces, would have for projections curves which would be remote from the traces themselves, and which would have to be constructed by means of points. If a system of planes parallel at the same time to the generating lines of the two surfaces be selected, each of these planes will intersect the two surfaces in straight lines, and these lines will intersect in points which are situated in the intersection of the two surfaces. In this way, then, the points of the horizontal projection will be constructed by means of the intersections of straight lines; a necessary consequence of what has been said respecting the case of two conical surfaces.

56. Lastly, for two surfaces of revolution whose axes are in the same plane, but not parallel to each other, instead of a system of planes, it would be better to select a system of spherical surfaces having the point of intersection of the two axes for their common center; for each of the spherical surfaces would cut the two surfaces of revolution in the circumferences of two circles

whose centers would be in the respective axes, and whose planes would be perpendicular to the plane drawn through the two axes; and the points of intersection of these two circumferences, which would lie at the same time both in the spherical surface, and in the two surfaces of revolution, would belong to the required intersection. Thus, the points of the projection of the intersection would be constructed by the intersections of circles and straight lines. In this case, the most advantageous position for the two planes of projection is for one of them to be perpendicular to one of the axes, and for the other to be parallel to the two axes. These few remarks respecting the curved surfaces, which are most frequently met with, suffice to show the manner in which the general method should be employed, and how, by the knowledge of the generation of the curved surfaces, to select that kind of section which will afford the simplest constructions.

ON TANGENTS TO THE INTERSECTIONS OF SURFACES.

57. When two curved surfaces have their respective forms and positions defined, not only is the curve of their intersection determined in space, but all the properties of this curve immediately follow. Thus, for example, at each of its points, the direction of its tangent is determined. So also is that of its normal plane, that is to say, of the plane which cuts the curve at right angles, and is, consequently, perpendicular to the tangent at the point of intersection. Although we shall often have occasion in the sequel to consider the normal planes to curves of double curvature, we will not enter now upon any detail respecting their determination, because these planes being always perpendicular to the tangents, it will suffice to have given the method for constructing the projections of the tangents to the intersections of curved surfaces.

58. Prob. 22.—Second General Prob.—Through any point in the intersection of the two curved surfaces, to draw a tangent to this intersection.

SOLUTION.—The point taken in the intersection of the two surfaces is situated at the same time in each of these surfaces. If, then, through this point, considered as a point in the first surface, a tangent plane to this surface be drawn, this plane will also touch the intersection in the point under consideration. In like manner if through the same point, considered as a point in the second surface, a tangent plane to this surface be drawn, this plane will also touch the intersection in the point under consideration. The two tangent planes will, therefore, touch the intersection at the same point, which is common to both planes, and which will therefore lie in the straight line in which they intersect; consequently, the intersection of the two tangent planes will be the tangent required.

This problem leads to the following observation which is of great use in descriptive geometry.

"The projection of the tangent to a curve of double curvature is itself a tangent to the projection of the curve, and its point of contact is the projection of that of the curve of double curvature."

In fact, if from every point of a curve of double curvature, perpendiculars be conceived to be let fall upon one of the planes of projection, for instance, upon the horizontal plane, all these perpendiculars will lie in a vertical cylindrical surface, whose intersection with the horizontal plane will be the horizontal projection of the curve. In the same manner, if through every point of the tangent to the curve of double curvature verticals be conceived to be drawn, they will lie in a vertical plane, whose intersection with the horizontal plane will be the horizontal projection of the tangent. Now it is clear that the cylindrical surface and the vertical plane touch each other throughout the whole length of the vertical let fall from the point of contact, which is common to both; and, therefore, the intersections of the cylindrical surface and of the vertical plane with the horizontal plane will touch each other in the point in which the line of contact of the cylindrical surface and vertical plane meets the plane of projection. Finally,

then, the projections of a curve of double curvature, and of one of its tangents, touch each other in a point which is the projection of the point of contact of the curve.

THE INTERSECTION OF CYLINDRICAL AND CONICAL SURFACES. THE DEVELOPMENTS OF THESE INTERSECTIONS, WHERE ONE OF THE SURFACES TO WHICH THEY BELONG, IS DEVELOPABLE. (pls. VI. to X., fig. 26 to 35).

59. We proceed to apply the preceding propositions to some particular cases, and, to commence with the most simple, we shall begin with cases in which one of the two surfaces whose intersection is to be determined is a plane.

PROB. 23.—To construct the intersection of a given cylindrical surface with a plane given in position.

The position of the planes of projection being arbitrary, we will first suppose that these two planes, as is always possible, have been so selected, that one of them may be perpendicular to the generating line of the surface, and the other perpendicular to the cutting plane; because the construction is much simplified by this supposition; and afterwards, to give students practice in the methods of projections, we shall suppose the two planes of projection to be placed in any manner whatsoever.

SOLUTION.—*Case* 1. When the generating line of the surface is supposed to be perpendicular to one of the planes of projection, for example, to the horizontal plane, and the cutting plane to be perpendicular to the other.

Let A (pl. VII. fig. 27) be the horizontal projection of the straight line to which the generating line of the cylindrical surface is always to be parallel; $a\,a'$ its vertical projection; B C D E the given trace of the cylindrical surface, which will be the horizontal projection of the surface in all its indefinite extent, and, consequently, that of the curve of intersection; let $f\,g$ be the given vertical projection of the cutting plane, a projection which will also be that of the required intersection, and

F G the horizontal trace of the same plane: it is evident that, if E e'' C c'' be drawn touching the curve B C D E and at right angles to L M, the straight lines $e\,e''$, $c\,c''$, will be the vertical projections of the generating line in its extreme positions, and that the points e', c', in which they cut the projection fg of the cutting plane, will be the limits of the vertical projection of the required intersection.

This being established, if through any point whatever in the intersection, a point whose horizontal projection is any point, H, in the curve B C D E, and whose vertical projection is found by projecting the point H to the point i on fg, it is required to draw the tangent to this intersection, it is clear that this tangent will be contained in the cutting plane, and that the straight line fg will be its vertical projection, it is also clear that it will be contained in the vertical tangent plane to the cylindrical surface, and that its horizontal projection, which will be the same as that of the tangent plane, will be the straight line F H N, touching the given curve B C D E at the point H. Thus everything regarding the required intersection has been determined.

60. Suppose now it were required to construct this intersection exactly as it lies in its own plane, and through one of its points, taken at pleasure, to draw a tangent to it. If the vertical plane of projection happens to be at too great a distance from the curve B C D E, another vertical plane can be imagined parallel to it, passing through the interior of the curve B C D E, whose horizontal projection may be the straight line E C parallel to L M. This vertical plane will intersect the cutting plane in a straight line parallel to its projection fg, and about which, as upon an hinge, let the cutting plane be conceived to revolve until it becomes vertical, and presents a front view of the required curve. This being done, let us conceive planes perpendicular at the same time to both planes of projection to pass through as many points H as we please, taken arbitrarily in B C D E, the horizontal and vertical projections of which planes will be obtained both at once by drawing through all

the points the straight lines H J K *i i'* perpendicular to L M. Each of these planes will intersect the cutting plane, in a horizontal straight line perpendicular to the hinge, and whose vertical projection will be the point, *i'*, in which the two straight lines *f g*, *i i'* meet each other. Moreover, in each plane, this horizontal straight line will meet the hinge in a point of which the intersection, J, of the two straight lines E C, H J K *i i'* will be the horizontal projection; it will meet the required curve in the points whose horizontal projections will be the intersections H, K, of the straight line H J K *i i'* with the curve B C D E; and lastly, both the entire line, and all its parts will be equal to their horizontal projections. Now, when the cutting plane revolves about the hinge in order to take a vertical position, all these straight lines, which at first were horizontal, continue to be perpendicular to the hinge, and of the same magnitude. If, therefore, through all the points *i'* straight lines, *h k*, of indefinite length, be drawn perpendicular to *f g*, and, if upon these perpendiculars J H be set off from *i'* to *h*, and J K from *i'* to *k*, as many points *h*, *k*, as we please will be obtained, through which the required curve *e' k c' h* is to be drawn.

61. The curve being constructed in its plane, if it is required to draw a tangent to it through, *h*, any point in it whatever, the vertical projection of this point will be obtained by letting fall from the point *h* upon *f g* the perpendicular *h i'*, and its horizontal projection by projecting *i'* to the point H upon the curve B C D E; the horizontal projection of the required tangent will be obtained by drawing the tangent F N to the curve B C D E at the point H; and it will suffice to trace back to the plane of the curve any second point whatever of the tangent, that, for instance, which is projected upon the point N, taken arbitrarily, and whose vertical projection is at the point *a'* in *f g*. Now, by reasoning with respect to this point in the same manner as for every other point of the cutting plane, it is clear that, if through the point *a'* we draw *a' n* perpendicular to *f g*, and upon

this line set off from a' to n the distance, N A, from the point N to the straight line E C, the point n will be a second point in the tangent. Drawing, then, the straight line $h\,n$, the required tangent will be obtained.

62. Whatever the given curve B C D E may be, we perceive that the intersection $e'\,k\,c'\,h$ has the property, that for any one of its points the sub-tangent $a'\,n$ is equal to the sub-tangent A N of the first curve. This property, which is well-known with regard to the ellipse and circle, when these two curves have a common axes, belongs to them only because they are intersections of the same cylindrical surface by two different planes.

63. Finally it may be required to trace on the development of the cylindrical surface the line of section made by the cutting plane. For this purpose, after having developed the curve B C D E with all its divisions into a straight line R Q, if through all the divisions of R Q perpendiculars to it of indefinite length be drawn, the traces of the different positions of the generating line upon the development of the surface will be obtained, and it will only remain to set off on these perpendiculars the portions of the corresponding generating lines, comprised between the perpendicular section B C D E, and the section made by the cutting plane. Now these portions of the generating lines are equal to their vertical projections, and these projections are all terminated on one side by the straight line L M, and on the other, by $f\,g$. Thus if the point H, for example, falls on the line R Q at S, setting off $i\,i'$ on the perpendicular which passes through S, from S to T, the point T will be on the developed surface that in which the generating line which passes through the point H, is intersected by the cutting plane. The curve X T Y Z, which passes through all the points determined in the same manner will be the curve required.

64. It is evident that, if the tangent at the point H be produced until it meets G F the horizontal trace of the cutting plane in a point F, and if H F be set off from S to U along R Q, the straight line T U will be a tangent to the curve; for when the

cylindrical surface is developed, its elements undergo no change in their inclination to the horizontal plane.

Second Case, in which we suppose the Cylindrical Surface and the cutting plane placed in any manner whatever with respect to the two planes of projection.

65. SOLUTION.—Let A A′ and *a a′* (pl. VII. fig. 28) be the two projections of the straight line to which the generating line is to be parallel; C E D F the given trace of the cylindrical surface; and H G, *h b* the traces of the cutting plane.

Imagine a series of planes parallel to the generating line of the cylindrical surface, and which shall moreover be all perpendicular to one of the planes of projection, to the horizontal plane for instance: these planes will be projected along straight lines O K E parallel to A A′, and will cut the surface in straight lines which will be positions of the generating line, and which will meet the horizontal plane at the points of intersection, E, F, of the straight line O K E with the curve C E D F. If then the points E, F, be projected to *e*, *f*, upon L M, and through these last points *e e′*, *f f′*, be drawn parallel to the straight line *a a′*, we shall obtain the vertical projections of the intersections of the surface with each of the planes parallel to the generating line.

These planes will also intersect the cutting plane in straight lines, which will be parallel to each other, which will have all their horizontal traces upon the different points O of the straight line H G, and of which the vertical projections will also be parallel to each other.

To obtain these projections, the direction of some one of them must first be found, of that, for instance, which corresponds to the vertical plane passing through A A′. For this purpose, if A A′ be produced until it meets, on one side, the trace of the cutting plane at the point N, and, on the other, the straight line L M at the point B, and if the point B be projected to *b* upon *h b*, the two points N, and *b*, will be the traces upon the two planes of projection of the intersection of the cutting plane with the vertical

plane; and if, then, the point N be projected to n upon L M, and the straight line $n\,b$ be drawn, we shall have the vertical projection of this intersection. Therefore, by projecting upon L M all the points, O, in which the trace G H is cut by the projections of the vertical planes, which will give a series of points o, and by drawing through these last the straight lines $o\,i\,k$ parallel to $n\,b$, we shall have the vertical projections of the intersections of the cutting plane with the series of vertical planes. Therefore, finally, the points of meeting i, k, of each straight line $o\,i\,k$, with the projections $e\,e'$, $f\,f'$, of the sections made in the cylindrical surface by the corresponding vertical plane, will be upon the vertical projection of the required intersection; and the curve passing through all the points i, k, thus determined will be this projection. If the points i, k be projected to J, K, upon the projection, O K E, of the corresponding vertical plane, we shall have the horizontal projection of the same points; and the curve K J P, passing through all the points thus determined, will be the horizontal projection of the intersection.

66. In order to obtain the tangents to these two projections at the points J, i, it must be borne in mind that these tangents are the projections of the tangent to the intersection. Now, this last tangent, being at the same time in the cutting plane, and in the tangent plane to the cylindrical surface, must have its horizontal trace in the intersection of the horizontal traces of these two planes: moreover, the trace of the tangent plane is the tangent to the curve C E D F at the point F. If, then, this tangent be drawn, and if, after having produced it until it meets the trace of the cutting plane in a point G, we draw the straight line G J, this straight line will touch the horizontal projection of the intersection in the point J; and by projecting the point G to g on L M, and drawing the straight line $g\,i$, we shall have the tangent at i to the vertical projection of the same curve.

67. If it were required to construct the curve of the intersection, as it exists in its own plane, conceive the cutting plane to revolve about its horizontal trace H G, as upon a hinge, until

it coincides with the horizontal plane. During this motion, each of the points of the section, for instance, that which is projected in J will describe the arc of a circle, whose plane will be vertical, perpendicular to H G, and have for its indefinite projection the straight line, R J S, drawn through J perpendicular to H G; so that, when the plane is thrown down, the point of the section will fall somewhere on this straight line. It remains to find the distance of this point from the axis of revolution. Now, the horizontal projection of this distance is J R, and the difference of the heights of its extremities is the vertical *i s*. If, then, J R be set off upon L M from *s* to *r*, the hypotheneuse *r i* will be this distance. By setting off, therefore, *r i* upon R J from R to S, the point S will be one of the points of the intersection, as it would appear in its own plane, when thrown down upon the horizontal plane; and the curve S T U V, drawn through all the points S similarly constructed, will be this intersection itself.

68. In order to obtain the tangent of this curve at the point S, it is sufficient to observe, that during the movement of the cutting plane, the tangent always passes through the point G of the axis of revolution; and, therefore, by drawing the straight line S G, we shall have the tangent required.

69. Prob. 24.—To construct the intersection of a conical surface upon any given base, with a plane given in position.

Solution.—Suppose the vertical plane of projection to be placed perpendicularly to the cutting plane, as it always can be.

Let A and *a′* (pl. VIII. fig. 29) be the projections of the vertex of the cone, or of the center of the conical surface; B C D E, the trace of this surface upon the horizontal plane; *f g*, the vertical projection of the cutting plane, and G *f* its horizontal trace. Imagine a succession of planes perpendicular to the vertical plane of projection to pass through the vertex of the cone: the vertical projections of these planes will be the straight lines *a′ c* drawn through the projection of the vertex; and their horizontal traces will be the straight lines *c* C perpendicular to L M, which will cut the trace of the conical surface somewhere in the points C, C′.

These planes will cut the surface in straight lines whose vertical projections will be the straight lines $a' c$, and whose horizontal projections will be obtained by drawing to the point A, the straight lines C A, C′ A. The same planes will also intersect the cutting plane in straight lines which will be perpendicular to the vertical plane. The vertical projections of these straight lines will be the points, h, in which fg meets the straight lines $a' c$, and their horizontal projections will be obtained by letting fall from the points h, on L M, the perpendiculars h H, of indefinite length. This being done, the straight lines h H will cut the corresponding straight lines C A, C′ A, &c., in the points H, H′, which will be the horizontal projections of as many points of the intersection required; and the curve P H Q H′, passing through all the points constructed in this manner, will be the projection of the intersection.

70. In order to draw a tangent to this curve through a point H, taken in it at pleasure, it is sufficient to find the trace on the horizontal plane of the tangent to the intersection at the point which corresponds to the point H. Now this trace must be on that of the cutting plane, and consequently, upon Gf; it must also be on that of the plane which touches the conical surface in the straight line whose projection is A H; and, moreover, if A H be produced until it meets the curve B C D E in some point C, the tangent C F, of this curve at the point C, will be the horizontal trace of the tangent plane. Therefore the point F, in which the two traces f G, C F meet each other, will be a point in the tangent to the curve P H Q H′ at the point H.

71. If it be required to construct the intersection as it would appear in its own plane, it may be supposed, either that the cutting plane is turned about Gf, as upon a hinge, so as to be thrown down upon the horizontal plane, and the curve constructed in the position which it then takes, or that it revolves about its vertical projection fg, so as to be thrown back upon the vertical plane; we proceed to adopt this latter hypothesis.

All the horizontal lines in which the succession of planes

drawn through the vertex have intersected the cutting plane, and which are perpendicular to fg, retain the same magnitude during the movement of the cutting plane, and remain constantly perpendicular to fg: so that, if through all the points h perpendiculars to fg be drawn of indefinite length, and if upon them the corresponding horizontal lines K H, K H′, be set off from h to N and to N′, the points N and N′ will be points of the section, and the curve R N S N′, drawn through all the points thus constructed, will be the intersection, as it would appear in its own plane.

72. From the preceding articles, it is evident that a tangent to this curve at a point N, taken arbitrarily upon it, is to be drawn by letting fall from the point N upon fg the perpendicular N h, drawing the straight line a' h, and producing it, until it meets L M in a point c, projecting this last point to C upon the curve B C D E, drawing the tangent to this curve at C, which will cut the trace Gf somewhere at a point F, and setting off Ff perpendicularly to fg from f to O. The straight line O N will be the tangent required.

With respect to the manner of constructing the development of the conical surface having any base whatever, and tracing upon this development the line of section made by the cutting plane, it will be investigated immediately after having treated of the intersection of the conical surface with that of a sphere having its center at the vertex.

73. PROB. 25.—To construct the intersection of two conical surfaces having circular bases, and whose axes are parallel to each other.

SOLUTION.—We will not repeat here the processes already gone through with fig. 26., pl. VI., in explaining the general method for which this figure served as the type, observing only that in the case now under consideration, as in that of any two surfaces of revolution, the sections made in the two surfaces by the horizontal planes are circles; but proceed to enter upon some details respecting the tangents not yet discussed.

74. In order to find the tangent at the point D (pl. VI., fig. 26) to the horizontal projection of the intersection, it is to be borne in mind that it is the projection of the tangent to the intersection of the two surfaces at the point which corresponds to D, and that, to determine it, it is sufficient to find the point S which is, on the horizontal plane, the trace of the tangent to the intersection. Now this last tangent lies in the two planes which touch the conical surfaces at the point of the intersection; so that the intersection of the horizontal traces of these two tangent planes will determine the point S. But the tangent plane to the first surface touches it in a straight line, which passes through the vertex, and whose horizontal projection is found by drawing the straight line A D of indefinite length. Moreover, if A D be produced, until it meets in a point Q the circular horizontal trace T Q U V of the surface, the point Q will be a point in the line of contact of the surface and the plane; and, consequently, the horizontal trace of the plane will be Q *q*, the tangent to the circle T Q U V at the point Q.

In like manner, if B D be produced until it meets in R the circular horizontal trace R X Y Z, of the second surface, and if the tangent to this circle at the point R be drawn, this tangent R *r*, will be the horizontal trace of the tangent plane to the second surface. If, therefore, the straight line S D be drawn through the point S of the intersection of the two tangents Q *q*, R *r*, we shall have the tangent at the point D to the horizontal projection of the intersection.

With regard to the tangent to the corresponding point *d* of the vertical projection, it is evident that it will be obtained by projecting the point S to *s*, and by then drawing the straight line *s d*, which will be this tangent.

75. It may happen to be required to construct upon the development of one of the conical surfaces, perhaps even upon that of each of them, the line of their mutual intersection; this would be necessary, for example, if we had to construct the cones of flexible substances, such as sheets of metal. In this case, we

should go through for each cone the steps we proceed to point out for the first.

We remark in the first place, that when a conical surface is developed so as to become a plane, the straight lines which are on this surface do not alter either in form or magnitude, since each of them is successively the axis abont which the development takes place; and thus all the points of the surface remain always at the same distance from the vertex. Moreover, when, as in this case, the surface is that of a right cone upon a circular base, all the points of the horizontal circular trace are at equal distances from the vertex; and they ought, therefore, to be at an equal distance from the vertex, upon the development, and, consequently, upon an arc of the circle of which the radius is equal to the constant distance of the vertex from the circular trace. If, therefore, after having taken any point to represent the vertex upon the development, an arc of indefinite extent be described from this point as a center, and with a radius equal to *a* C, this arc will also be indefinitely the development of the horizontal trace of the surface. If, then, setting out at the point T, of the trace, from which we wish the development to commence, the arc of the circle T Q be measured off upon the arc which we have just described, the position of the point Q upon the development will be determined; and the indefinite straight line, drawn through this point to the center of the development, will be the position which will be occupied by the straight line of the surface, of which A Q is the projection, and upon this line the point D, *d* of the section, when reierred to this development, will be found.

To construct this point, it only remains to find its distance rom the vertex, and to measure it off upon the indefinite straight line, setting out from the center of the development. For this purpose, through the point *d* in the vertical projection, draw the horizontal line *d k* until it cuts the side *a* C of the cone in a point *k;* and the straight line *a k* will be this distance. By constructing successively in the same way all the other points of the inter-

section, and drawing a curve through all these points, we shall obtain the intersection of the two surfaces referred to the development of the first; and the same process must then be gone through for the second surface.

76. Prob. 26.—To construct the intersection of two conical surfaces having any bases whatever.

Solution.—Let A, a (pl. XIII. fig. 30) be the projections of the vertex of the first surface, C G D G′ its given trace upon the horizontal plane, B b, the projections of the vertex of the second surface, and E H F H′ its trace upon the horizontal plane. Imagine a straight line to pass through the two vertices, whose projections will be obtained by drawing the indefinite straight lines A B, $a\,b$, and of which the trace, I, upon the horizontal plane, can be readily constructed. Conceive a series of planes to pass through this straight line, each of which will cut the two surfaces in a system of several straight lines; and of these straight lines those lying in the same plane will determine by their meeting, so many points of the intersection of the two surfaces. The horizontal traces of all the planes of this series will necessarily pass through the point I; and because the position of these planes is in other respects arbitrary, their traces can be constructed arbitrarily, by drawing through the point I as many straight lines I K as may be convenient, with each of which the same operation is to be gone through, as we proceed to describe for one of them.

The trace K I of any one of the planes of the series will cut the horizontal trace of the first conical surface in the points G, G′, which will be also the horizontal traces of the straight lines in which the plane cuts the conical surface; so that A G, A G′, will be the indefinite horizontal projections of these lines; and their vertical projections will be obtained by projecting G, G′ to g, g', and drawing the indefinite straight lines $a\,g$, $a\,g'$. In like manner the same trace K I will cut the horizontal trace of the second conical surface in the points H, H′; and the straight lines B H, B H′, drawn through these points, and produced indefinitely, will be the horizontal projections of the straight lines in which the

same plane of the series cuts the second surface; and their vertical projections will be obtained by projecting H, H′, to h, h', and drawing the indefinite straight lines $b\,h$, $b\,h'$.

This being done, for the same plane whose trace is K I, we shall have on the horizontal plane of projection a certain number of straight lines; and the points P, Q, R, S, in which those belonging to one of the surfaces meet those belonging to the other, will be the horizontal projections of as many points of intersection of the two surfaces. Now, operating in the same manner with others of the lines K I, successively, new sets of points P, Q, R, S, will be found; and then making one branch of a curve pass through all the points P, a second through all the points Q, a third through all the points R, &c., the horizontal projection of the required intersection will be obtained.

In like manner, for the same plane of which the trace is K I, we shall have on the vertical plane of projection a certain number of straight lines $a\,g$, $a\,g$, $b\,h$, $b\,h'$, meeting one another in points which will be the vertical projections of as many points of the intersection.

It is to be observed here that it is not necessary to construct the two projections of the curve of intersection independently of each other; but that any point of one of them being constructed, the corresponding point of the other projection can be found by projecting it perpendicularly to the common intersection of the two planes of projection, on the one of the straight lines on which it must be situated: a method which furnishes the means of verifying the work, and of avoiding, in certain cases, the errors of practice in obtaining points from the intersections of lines which cut at angles too oblique.

77. To find the tangents to the horizontal projection, that, for example, which touches it at the point P, the horizontal trace, T, of the tangent to the intersection at the point which corresponds to P, must be constructed. Now this tangent being the intersection of the two planes which touch the conical surface at that point, its trace will be the intersection of the horizontal

traces of these two planes. Moreover, A G′ P is the projection of the line of contact of the plane which touches this first surface; so that the trace of this tangent plane will be the tangent to the curve C G D G′ at the point G′: let G′ T V be this tangent. In like manner B H′ P is the horizontal projection of the line of contact of the plane which touches the second surface; so that the horizontal trace of this second tangent plane will be the tangent to the curve E H F H′ at the point H: let H′ T U be this tangent. The two tangents G′ V, H′ U will intersect then in a point T, through which drawing the straight line T P, the required tangent at the point P will be obtained.

By reasoning in the same way for the other points Q, R, S, it will be found, (1.) that the tangent at Q must pass through the intersection of the tangents at G′ and at H; (2.) that the tangent at R must pass through the intersection of the tangents at H and at G; (3.) that the tangent at S must pass through the intersection of the tangents at G and at H′.

With regard to the tangents to the vertical projection there will be no difficulty, when those to the horizontal projection have been determined; for, by projecting the horizontal traces of the tangents to the intersection, points will be found through which they must pass.

78. Prob. 27.—To construct the intersection of the surface of a sphere with that of a cone having any base whatever.

We shall suppose here that the two surfaces are concentric, that is to say, that the vertex of the cone is placed at the center of the sphere, because this particular case will be required for the succeeding problem.

Solution.—Let A and *a* (pl. IX., fig. 31) be the projections of the common center of the two surfaces, B C D E the given horizontal trace of the conical surface, *a m*, the radius of the sphere, and the circle *l f′ g′ m* the vertical projection of the sphere. Conceive a series of planes all perpendicular to one of the planes of projection to pass through the common centre of the two surfaces: in fig. 31, these planes have all been con-

ceived vertical. Each of these planes will cut the conical surface in a system of straight lines, and the surface of the sphere n the circumference of one of its great circles; and the intersections of these lines with the circumference of the circle will be points of the required intersection. Let there be drawn, then, through the point A as many indefinite straight lines, C A E, as we wish, which will be the horizontal projections of as many of the series of vertical planes, and at the same time those of the lines in which these planes cut the two surfaces. Each straight line C A E will cut the horizontal trace, B C D E, of the conical surface in points C, E, which will be the horizontal traces of the sections of this surface made by the corresponding plane; and if after having projected the points C, E, to c, e, on L M, the straight lines, $a\,c$, $a\,e$, be drawn, the vertical projection of the same sections will be obtained. It remains to find the intersections of these sections with those of the sphere made by the same plane.

For this purpose, after having drawn through the point A the straight line G A F parallel to L M, let the vertical plane drawn through C E be imagined to turn about the vertical line which passes through A, and is projected to $a'\,a$, until it becomes parallel to the vertical plane of projection, and let it be supposed to carry with it the sections of the two surfaces made by it. In this movement, the points C, E, will describe about the point A, as a centre, circular arcs C G, E F, and will come into the positions G, F; and if these latter points be projected to g and f upon L M, the straight lines $a\,g$, $a\,f$, will be the vertical projections of the sections of the conical surface in the new position taken up by them in consequence of the movement of the plane. The section of the surface of the sphere in its new position will have for its vertical projection the circumference $l\,f'\,g'\,m$. The points of intersection, f', g', of this circumference with the straight lines $a\,g$, $a\,f$, will be, then, the projections of points of the required intersection, in the supposed new position of the plane.

Now to obtain the projections of the same points in their actual position, the vertical plane must be imagined to return to its original position. In this movement all the points of the plane, and consequently those of the intersection contained in it, will describe about the vertical line passing through A, as an axis, horizontal circular arcs, whose vertical projections will be horizontal straight lines. If, then, through the points $g' f'$, the horizontals $g' i$, $f' h$ be drawn, they will contain the vertical projections of points of the intersection; but these projections must also be situated upon the straight lines $a c$, $a e$, respectively; they will, therefore, coincide with the points i, h, in which these last straight lines meet the horizontals $g' i$, $f' h$. Thus the curve $k h n i$, drawn through all the points constructed in the same manner for every other straight line like A E, will be the vertical projection of the intersection required.

If the points i, h be projected to J, H, upon C E, the horizontal projections of the same points of intersection will be obtained; and the curve K H N J, drawn through all the points J, H, constructed in the same manner for every other line like C E, will be the horizontal projection of the intersection.

79. To find the tangent at the point J of the horizontal projection, the horizontal trace, P, of the tangent at the corresponding point of the intersection must be constructed. This trace ought to coincide with the intersection of the traces of the tangent planes to the two surfaces at the point of the intersection which corresponds to J. Now it is evident that, if through the point C the straight line C P be drawn to touch the curve B C D E, the trace of the tangent plane to the conical surface will be obtained. With respect to that of the tangent plane to the sphere, it will be constructed in the manner already explained for surfaces of revolution, that is to say, by drawing through the point g' a straight line, $g' o$, to touch the circle $l f' g' m$, and producing it to meet L M in o, by setting off, then, $a' o$ upon C E from A to O, and drawing through the point O the straight line O P perpendicular to C E. Then the two traces C P, O P, will

intersect in a point P, through which, if the straight line J P be drawn, the tangent at the point J will be obtained.

Lastly it is evident that the tangent at the point *i* of the vertical projection of the intersection will be obtained by projecting the point P to *p* upon L M, and then drawing the straight line *i p*, which will be the required tangent.

80. If the sphere and the conical surface had not been concentric, a straight line must have been conceived to pass through their two centers, and the series of cutting planes passing through this line must have been selected. Each of these planes would have cut the conical surface in straight lines, and that of the sphere in one of its great circles, as in the preceding case, which would have given an equally simple construction; but then it would have been advantageous to place the vertical plane of projection parallel to the line passing through the two centers, in order that, in the movement that each cutting plane is made to go through to become parallel to the vertical plane of projection, the two centers may remain stationary, and retain the same projections, which simplifies the constructions.

81. Prob. 28. To construct the development of a conical surface having any base whatever, and to set off on the surface thus developed a section, the two projections of which are given.

Solution. Conceive the surface of a sphere to be described with the vertex of the cone as a center, and any radius whatever, and construct, as in the preceding problem, the projections of the intersection of these two surfaces. This being done, it is evident that all the points of the spherical intersection being at the same distance from the vertex, they must also be at the same distance from the vertex on the developed surface, and will consequently be situated on an arc of the circle described from the vertex as center, with a radius equal to that of the sphere. Thus, supposing the point R to be the vertex of the developed surface, if from this point as center and with radius equal to *a m*, an arc, S T U, of a circle be described, all the

points of the spherical intersection will fall upon this arc in such a manner that the parts of the arc will be respectively equal to the corresponding parts of the spherical intersection. It remains, then, now, after having taken any point whatever on this intersection as origin, for example, that which is projected to N, *n*, and a point S for its corresponding point on the developed surface, to develope the different arcs of the spherical intersection, and to set them off successively on the circular arc S T U from S to different points T. For this purpose the spherical curve being a curve of double curvature, it must be made to lose successively its two curvatures, without changing its magnitude, in the following manner:—

The spherical intersection, being projected to N J K H on the horizontal plane, can be regarded as traced on the surface of a vertical cylinder having N J K H for its base: this surface can, then, be developed in the manner pointed out in Art. 63, and the spherical intersection can be set off on this developed cylindrical surface, by developing the arc N J (fig. 31) on N′J′ (fig. 32) and setting off the vertical *i′ i* at right angles to N′ N′ from J′ to J″. The curve N″ J″ K″ H″ N″, passing through all the points, J″, thus determined, will be the spherical intersection deprived of its horizontal curvature, without having changed in length. The tangent at the point J″ of this curve will be obtained by taking J P (fig. 31), setting it off on N′ N′ from J′ to P′, and drawing the straight line J″ P′.

The curve N″ J″ K″ H″ N″ is now to be developed by bending it back on the arc S T U (fig. 33); for instance, the arc N″ J″ is to be set off from S to T, and the point T will be the situation on the developed conical surface of that point of the spherical intersection whose projections are J, *i*. If, then, the straight line R T be drawn, we shall have, on the development of the surface, the generating line whose horizontal projection is A C (fig. 31). Lastly, to mark upon the developed surface a point situated on this generating line it is only required to take the distance of this point from the vertex of the conical surface,

and to set it off on R T from R to V; and the point V will be on the point desired, the developed surface.

82. PROB. 29.—To construct the intersection of two cylindrical surfaces having any bases whatever.

SOLUTION.—When, in the investigation which gives rise to the question under consideration, there are no other intersections to be taken into account but that of the two cylindrical surfaces, and especially when these surfaces have circular bases, it is advantageous to select the planes of projection so that one of them may be parallel to the generating lines of both cylinders: by which means the intersection is constructed without the employment of any other than the given curves. But, when the intersections of these surfaces with others are to be considered at the same time, it is not advisable to change the planes of projections but is more simple to represent the objects by referring them all to the same planes.

The generating lines of the two surfaces we shall, therefore, suppose to be situated in any manner whatever with respect to the planes of projection.

Let, then, T F F′ U, X G G′ V (pl. IX., fig. 34) be the given horizontal traces of the two cylindrical surfaces; A B, *a b*, the given projection of the straight line to which the generating line of the first surface must be parallel; and C D, *c d*, those of the straight line to which the generating line of the second must be parallel. Conceive a series of planes parallel to the two generating lines. These planes will cut the two surfaces in straight lines; and the points in which the two sections made in the first surface meet those made in the second, will determine the points of the required intersection.

Thus, after having constructed, as in Prob. 13, Art. 31, the horizontal trace A E of a plane drawn through the first given straight line parallel to the second, as many straight lines as you please are to be drawn parallel to this trace, and these parallels are to be regarded as the traces of the series of planes. Each parallel F G′ will cut the trace of the first surface in points

F, F′, and that of the second in other points G, G′, through which the straight lines F H, F′ H′, &c., G J, G′ J′, &c., are to be drawn parallel to the projections of the respective generating lines; and the points P, Q, R, S, in which these lines meet, will be the horizontal projections of as many points of the intersection of the two surfaces. By going through the same steps with each of the set of lines F G′, a series of systems of points P, Q, R, S, will be found, and the curve passing through all the points found in this manner will be the horizontal projection of the intersection.

To obtain the vertical projection, the points F, F′, &c., G, G′, &c., are to be projected to *f*, *f*′, &c., *g*, *g*′, &c., upon L M, and through these latter points *f h*, *f*′ *h*′, &c., *g i*, *g*′ *i*′, &c., are to be drawn parallel to the projections of the respective generating lines, and the points in which these parallels meet will determine the vertical projections *p*, *q*, *r*, *s*, of points of the intersection. Going through the same steps with all the other lines F G′, new points *p*, *q*, *r*, *s*, will be found; and the curve passing through all these points will be the vertical projection of the intersection.

To obtain the tangents to these curves at the points P and *p*, construct first the horizontal trace F′ Y of the tangent plane to the first cylindrical surface, at the point of which these points are the projections; then the trace G′ Y of the tangent plane at the same point to the second surface; and the straight line drawn from the point P to Y, the point of intersection of these traces, will be the tangent at P. Lastly, projecting Y to *y* upon L M, and drawing the straight line *p y*, the tangent at the point *p* of the vertical projection will be obtained.

83. Prob. 30.—To construct the intersection of two surfaces of revolution, the axes of which are in the same plane.

Solution.—Let the planes of projection be so placed that one of them may be perpendicular to the axis of one of the surfaces, and that the other may be parallel to both axes. After this, let A (pl. X., fig 35) be the horizontal projection of the axis of the first surface, *a a*′ its vertical projection, and *c d e* the given generating line of this surface. Let A B parallel to L M

be the horizontal projection of the axis of the second surface, *a′ b* its vertical projection, so that A and *a′* are the projections of the point of intersection of the two axes; and let *f g h* be the given generating line of this second surface. Conceive a series of spherical surfaces, having their common center at the intersection of the two axes. For each of the surfaces of this series, construct the projection, *i k n o p q*, of the great circle parallel to the vertical plane of projection; and these projections, which will be circular arcs, described from the point *a* as center, and with arbitrary radii, will cut the two generating lines in points *k*, *p*.

This being premised, each spherical surface will cut the first surface in the circumference of a circle, the plane of which will be perpendicular to the axis *a a′*, the vertical projection of which will be obtained by drawing the horizontal *k o*, and its horizontal projection by describing with the point A as center, and with a diameter equal to *k o*, the circumference of a circle K R O R′. Moreover, each of the series of spherical surfaces will cut the second surface of revolution in the circumference of a circle, the plane of which will be perpendicular to the vertical plane of projection, and of which the vertical projection will be obtained by drawing through the point *p* a straight line *p n* perpendicular to *a′ b*.

If the point *r* in which the two straight lines *k o*, *p n* intersect, is nearer the two axes, respectively, than are the points *k*, *p*, it is evident that the two circumferences will cut in two points, of which the point *r* will be the common vertical projection; and the curve drawn through all the points *r*, constructed in the same manner, will be the vertical projection of the intersection of the two surfaces. Projecting the point *r* to R and R′ on the circumference of the circle K R O R′, will determine the horizontal projections of the two points in which the circumferences, situated upon the same sphere, intersect; and the curve drawn through all the points R, R′, constructed in the same manner, will be the horizontal projection of the required intersection.

These examples will be found sufficient to make the student acquainted with the manner of employing the method of con-

structing the intersections of surfaces, and drawing their tangents, especially if he exercises himself in constructing them with the greatest precision on a large scale, and if he, as far as possible, traces out the curves to their full extent.

ROBERVAL'S METHOD OF DRAWING A TANGENT TO A CURVE WHICH IS GIVEN BY THE LAW OF THE MOTION OF A GENERATING POINT. APPLICATION OF THIS METHOD TO THE ELLIPSE, AND TO THE CURVE RESULTING FROM THE INTERSECTION OF TWO ELLIPSOIDS OF REVOLUTION, HAVING A COMMON FOCUS.

We have hitherto regarded curves of double curvature as determined, each, by two curved surfaces of which it is the intersection, and this is, in fact, the point of view in which they most ordinarily present themselves in descriptive geometry. In this case it has been seen to be always possible to draw tangents to them. But a curved surface can be defined as well by means of the form and of the motion of its generating line, and it may also happen that a curve may be given by the law of the motion of a generating point; and then to draw a tangent to it, without having recourse to analysis, Roberval's method may be employed. This method, which he invented before Descartes had applied algebra to geometry, is implicitly comprehended in the processes of the differential calculus, on which account it is not noticed in elementary mathematics; we shall in this place confine ourselves then to a general exposition of the method. Those who wish to see numerous applications of it may consult the "Memoires de l'Academie des Sciences" previous to 1699, in which the works ot Roberval have been collected.

When, according to the law of its motion, a generating point is perpetually pushed towards the same point in space, the line that it traverses in virtue of this law is a straight line; but if, at each instant of its motion, it is pushed towards two points at the same time, the line that it traverses, and which, in some particular cases, may still be a straight line, is generally a curve. The

tangent to this curve at any point will be obtained by drawing through this point two straight lines in the two different directions of the motions of the generating point, setting off on these lines, on whichever side of the point may be convenient, distances proportional to the two respective velocities of this point, completing the parallelogram, and drawing the diagonal, which will be the tangent required; for this diagonal will be in the direction of the motion of the generating point, at the point of the curve under consideration.

86. We shall only cite one example:—

A thread A M B (pl. X. fig. 36) being attached by its extremities to two fixed points A, B, if this thread be stretched by means of a point, and this point be moved in such a manner that the thread may always be in a state of tension, the point will describe a curve D C M, which will be, as we know, an ellipse of which the fixed points A, B, are the foci. By means of the generation of this curve, it is very easy to draw a tangent to it by the method of Roberval. In fact, since the length of the string is constant, the distance A M is lengthened at each instant of the motion by the same quantity as the distance B M is diminished. The velocity of the describing point in the direction A M is therefore equal to the velocity in the direction M Q. If, then, equal straight lines be cut off from M B, and from A M produced, and the parallelogram M P R Q be completed, the diagonal M R of this parallelogram will be the direction of the motion of the generating point at M, and consequently the tangent to the curve at this point. It is clearly seen from this, that in the ellipse the tangent bisects the angle B M P, formed by one of the focal distances and the production of the other; that the angles A M S and B M R are equal to one another, and that the curve must have the property of reflecting to one of the foci rays of light emanating from the other.

87. It is easy to extend Roberval's method to the case of three dimensions, and to apply it to the construction of the tangents to curves of double curvature.

In fact, if a generating point is moved in space in such a manner that at each instant of its motion it is pushed towards three different points, the line that it traverses, and which, in some particular cases, may be a plane curve or even a straight line, will be, in general, a curve of double curvature. The tangent to this curve at any point will be obtained by drawing through this point straight lines in the directions of the three different motions of the generating point, setting off on these straight lines on whichever side of the point may be convenient, distances proportional to the three respective velocities of this point, completing the parallelopiped, and drawing the diagonal which will be the tangent to the curve at the point under consideration.

ON TWISTED SURFACES.

88. All surfaces generated by the motion of a straight line are called ruled surfaces, and are divisible into two classes, viz., *Developable*, and *Twisted Surfaces*. In developable surfaces two successive generating lines always lie in one plane.

In twisted surfaces, on the contrary, no two successive generating lines lie in one plane; so that such surfaces cannot be developed, or folded back upon one plane, without tearing or doubling some portion.

Twisted surfaces can always be regarded as generated by a straight line of indefinite length, moving so as to keep constantly in contact with three given curves, which thus direct its motion. If a system of conical surfaces be conceived having their vertices on the first of the three given curves, and all passing through the second of these curves, the points in which these surfaces meet the third given curve will determine the different positions of the generating line. Let M E N, M′ E′ N′, M″ E″ N″, (pl. X. fig. 37), be the three given directing curves, the figure being drawn in perspective; and, having taken any point E on the first curve, the conical surface having this point for its vertex, and passing through the second curve M′ E′ N′, would cut the third

in a point E″, which would determine the position of the line E′ E″ on the conical surface. The position of this line will be that of the generating line corresponding to the point E; and in the same manner would be found the position of the generating line for each of the points of the curve M E N.

When the three directing lines are straight lines, the twisted surface generated possesses the very remarkable property that it can be generated by the same straight line in two different ways. The first mode of generation follows from the definition that is given of the surface; the generating line during its motion keeps constantly in contact with three given lines.

The second mode of generation is deduced from the first, as follows:—Among all the straight lines which touch at the same time the three given lines, any three selected at pleasure may be considered as new directing lines, and the generating line during its motion keeping always in contact with these three lines will generate the same surface as before.

89. Through a given point on a twisted surface to draw a tangent plane to this surface.

Let M E N, M′ E′ N′, M″ E″ N″, be the three directing curves which govern the motion of the generating line; Z the given point on the twisted surface; E, E′, E″, the points in which the generating line, when passing through Z, meets the directing curves; and E T, E′ T′, E″ T″, the tangents to the directing curves at the points E, E′, E″.

Conceiving the twisted surface of which the three tangents E T, E′ T′, E″ T″, would be the directrices, let E E′ E″, F F′ F″, G G′ G″, be any three positions of its generating line, and it is easily seen that this surface, and the given surface, have a common element, E E′ E″, throughout which they touch; so that every tangent plane to the one necessarily touches the other. Now, to draw through the given point Z a tangent plane to the twisted surface which has the three tangents for directrices, two straight lines, through which this plane must pass, must be determined. But it has been seen (88) that through each point Z of this sur-

face two straight lines can be traced upon it, the one, E Z E″, passing through the three tangents E T, E′ T′, E″T″, the other Z Y X, meeting the three elements E E′ E″, F F′ F″, G G′ G″. The plane then passing through the two straight lines E Z E″, Z Y X, is the tangent plane to the twisted surface in its most general form, at a given point Z of this surface.

This solution assumes that the method of drawing tangents to the directing curves is known: it has been shown how to draw the tangents to the curves when they result from the intersection of two known surfaces, or are given by the law of movement of a generating point.

Having drawn a tangent plane to a twisted surface through a given point in this surface, it might be proposed to resolve the inverse problem: "A tangent plane being given, to find the point of contact." The solution would be an easy deduction from the above.

In fact, the tangent plane to a twisted surface necessarily passes through one of the positions of the generating line. Let E E″ be the element through which the given tangent plane must pass. Produce this plane until it meets the straight lines F F′, G G″, and cuts them each in one point; the straight line drawn through the two points of intersection, will have, with the element E E″, a common point, which is the point of contact required.

When a plane touches a developable surface, the contact takes place throughout the entire length of the straight line common to the plane and to the surface; but when it touches a twisted surface, the contact takes place only in a single point of the line which is common to them; at every other point of this straight line it is a secant.

CHAPTER IV.

ON THE THEORIES OF SHADOWS AND PERSPECTIVE.

THEORY OF SHADOWS.

90. It has been stated, that descriptive geometry must be regarded under two points of view. Under the first, it is to be considered as a means of researches for arriving with precision at certain desired results; and it is thus that it is employed in stone cutting and carpentry. Under the second, it is simply a means of representing objects, and, in this case the determination of shadows is an auxiliary advantage to it.

Persons conversant with the methods of this science, are aware that a single projection does not suffice to define an object; that two projections are necessary; because on a plane one of the dimensions is always wanting, but, by means of two projections, the three dimensions are determined. In examining then the description of an object completely given by means of its two projections, the horizontal projection must be compared with the vertical projection; and it is from this perpetual comparison that the knowledge of the form of the proposed object is deduced.

Although the method of projections is simple, and possesses a peculiar kind of elegance, yet this obligation of comparing incessantly two projections one with the other, is a trouble, which can be considerably diminished by the employment of shadows.

Suppose, for instance, that we have a horizontal projection, containing all the dimensions in length and breadth, but which determines nothing respecting the dimensions in height; if the bodies be considered to be illumined in a known manner (and it is convenient to adopt, in general, the manner the most natural, that with which we are most familiar) for example, by parallel rays of light, these bodies will be throwing shadows on one another, and on the horizontal plane above which they are

placed, and by means of the extent and forms of these shadows, we can immediately judge of the vertical dimensions. Thus the direction of the rays of light being known, there is no need of two projections: one only, with the traces of the shadows, will give a complete idea of the object under consideration; and if we have both the horizontal and vertical projections with the shadows constructed, these two projections will be more easy to read, and will show the object more readily, than if we only had the bare projections without shadows.

Thus, for all the arts connected with the representation of objects, in which descriptive geometry is used as a means, not of investigation, but of description, the determination of shadows is advantageous, and renders more perfect the representation proposed to be drawn.

The determination of shadows comprises two distinct branches; the one is the graphic construction of the outline of the shadows; the other is the investigation of the intensity of the tints to be given to each part of the surfaces which receive these shadows.

We shall discuss first the branch of the subject which relates to the construction of the outline.

ON THE GRAPHICAL CONSTRUCTION OF SHADOWS, Pl. X. and XI. Figs. 38 and 39.

91. The theory of shadows is entirely founded on the well known phenomenon that light is propagated in straight lines. We are so accustomed to this proposition, that to verify the straightness of a line we compare it with a ray of light. The straightness of a rule is examined by comparing it throughout its entire length with the ray of light passing through its two extremities; to know if a row of trees is in line, we place ourselves so that the ray of light coming from one extremity of this row to the eye, passes along the trees, and if they are all placed exactly in contact with this ray, they are recognised to be perfectly in line.

It will be admitted then as a principle, that light spreads in straight lines. It must, however, be observed that this proposition is only strictly true when the medium in which the light moves is of an uniform density; but in the applications to the art here under consideration, it is seldom required to regard the rays of light as produced to a great distance, and traversing media of sensibly different densities; it will therefore be allowable to suppose the media uniform, and the rays of light strictly in a straight line.

We shall confine our attention to the case in which the illumination proceeds from a single luminous point, as this case includes nearly all that is practically useful in the arts.

The luminous point shoots forth in all directions rays of light which collectively fill the whole of space, if no body present itself in their direction to arrest their progress. When, however, an opaque body, that is one which the rays of light cannot penetrate, either stops them or reflects them entirely or partly, the rays which do not meet the body continue to spread into space; but those in the direction of which it is placed, will be arrested, and will not extend into the part of space which is beyond, and which, by the interposition of the body, will be thus deprived of light.

Conceive a conical surface, having its vertex at the luminous point, and enveloping the opaque body, and suppose it produced indefinitely; it will be, on the other side of the opaque body, the limits of the part of space into which will penetrate the rays sent forth by the luminous point, and of that into which none will enter. This last part, deprived of light by the interposition of an opaque body, is what is called the shadow of this body; such is at least the definition of what is to be understood by the term shadow, when speaking of an eclipse of the moon, for example, the moon is said to enter into the earth's shadow. The sun is the luminous body, whence the rays start forth and spread in every direction; the earth is the opaque body, which intercepts a portion of these rays; and the portion of space behind

her, with respect to the sun, is deprived of light. As long as the moon is without this portion of space, she is illumined, and reflects light, she is visible; but from the moment of her entering into it, she receives no more light, she reflects no more, and becomes invisible.

In ordinary language, however, the above is not that which is most often understood by the term shadow, when, for example, while walking in the sunshine, one remarks that the shadows are short at noon. In this expression, the shadow is not the space deprived of light by the interposition of a body which arrests a portion of the rays sent forth by the luminous point; but it is the projection of this space on the surface which receives it. It is in this latter sense, that we shall usually employ this word.

If the luminous point be at an infinite distance, the rays of light which come to us from it will be parallel, nearly as those of the sun appear to us. On this hypothesis, to which we shall first direct our attention, two cases present themselves, that in which the opaque body which casts the shadow is bounded by plane surfaces, and, consequently, by rectilinear edges, and by vertices of solid angles, and that in which it is bounded by curved surfaces. The former case, which is extremely simple, will be first considered.

If the body which receives the light and casts the shadow is bounded by plane faces, it is easily comprehended that one portion of these faces is illumined, that the other is in shade, and that the line on this body, which separates the part illuminated from that which is not so, is formed by the rectilinear edges taken together, which are the intersections of the dark faces with the bright faces; this line is easily found, and determines the outline of the required shadow. If the opaque body be imagined to be removed, while this line, supposed to have a sensible thickness, continues in its place, the shadow of this line, traced on the surface which receives it, will be the outline of the shadow of the body. It is seen, then, that in the case under

consideration, the problem is reduced to that of finding the shadows of certain straight lines known in position.

To fix the ideas, and render the preceding remarks more tangible, suppose the body which casts the shadow to be the parallelopiped A B C D *a b c d* (pl. X. fig. 38), the direction of the parallel rays of light to be indicated by L *l*, and the plane M N to be the surface which is to receive the shadow. It is seen immediately, from the direction of the rays of light, that the faces A B C D, A B *a b*, A D *a d*, are illuminated, and that the faces D C *d c*, C B *c b*, and *a b c d*, are not so; that the edges D C, C B, B *b*, *b a*, *a d*, and *d* D are the boundaries of the bright and dark portions. The shadows D′ C′, C′ B′, B′ *b′*, *b′ a′*, *a′ d′*, and *d′* D′ of these six edges, on the plane M N form the outline or the boundaries of the shadow of the parallelopiped; the shadows of the six other edges falling in the interior of the area enveloped by this outline, are confounded in the complete shadow of the proposed body.

In general, when bodies bounded by plane surfaces are under consideration, the limiting edges, or those which separate the bright from the dark faces, are immediately distinguished, or easily determined; and a simple means of knowing them with certainty will be hereafter pointed out, if in any case their position should appear to admit of uncertainty. The question is confined then, as has been stated above, to the determination of the shadow of a certain assemblage of straight lines known in position.

To find in the first place the shadow of one of these lines. It will be observed that the body which casts the shadow being known in form and position with respect to the planes of projection, the edges which bound its faces are equally known with respect to these same planes, that is to say, their horizontal and vertical projections are either given or can be found. Supposing the luminous body to be a single point situated at an infinite distance, the direction of the rays of light, in this case, will be given by the horizontal and vertical projections of a straight line

to which they are all parallel. The rays of light meeting the straight line, whose shadow is to be determined, form a plane given in position, with respect to the planes of projection, by the condition of passing through the proposed straight line and being parallel to the direction of the light. This plane, produced, evidently contains the shadow of the line; if the body be considered, of which this line is one of the edges, it separates the illumined portion of space from that which the interposition of this body deprives of light. This same plane meets the surface on which the shadow is received in a certain line, which is the shadow cast by the straight line on this surface, or which belongs to the outline of the shadow of the given body. The surface being known and determinate with respect to the planes of projection, its intersection with the plane thus conceived can always be constructed, and thence this part of the outline of the required shadow can be completely determined.

The process thus gone through for one of the edges of the body which casts the shadow, can be repeated for a second edge, for a third, and in a word, for all those which together form, on this body, the division between the bright and dark faces.

If the luminous point were at a finite distance, the preceding solution would be still applicable with the introduction of a slight modification. The rays of light starting from this point of which the projection must be known, and directed towards the first of the edges to be considered, will form as before a plane determinate in space, or with respect to the planes of projection, from the condition of passing through this edge and the luminous point; and the consequences that have been deduced above with regard to the plane which, in the former hypothesis, contained the parallel rays of light, will again follow for that which contains the rays, when they are sent forth from a point situated at a finite distance.

It appears, then, that these investigations are but simple applications of the methods of descriptive geometry. To find on the body which casts the shadow, the edges which divide the

bright portion from the dark; through these edges to draw planes parallel to the direction of the rays of light, or containing the luminous point, if it is not an infinite distance; and to construct the intersections of these planes with the surface which is to receive the shadow: in the case under consideration, such is the entire solution.

It has been said, that the limiting edges, whose shadows circumscribe the shadow of the body, are, in general, easily to be distinguished; and, in fact, it is sufficient for this purpose to find the shadows of all the edges indifferently: those of them which enter into the interior of the polygon forming the outline of the shadow of the body, cannot belong to the limiting edges. Thus, in fig 38, the shadows $b' c'$, d', c', $C' c'$, $A' a'$, $A' D'$, $A' B'$, of the edges $b c$, $d c$, $C c$, $A a$, $A D$, $A B$, do not belong to any of the limiting edges, since they enter into the interior of the polygon $a' b' B' c' D' d'$.

But it can be determined with less trouble, whether of any two plane faces of a body one is bright and the other dark, or whether both are dark, or both bright, and consequently, whether their intersection is a limiting edge or not. Through any point of this intersection, imagine a ray of light: if of the two faces, one is bright and the other dark, this ray of light produced will leave them both on the same side; but if they are both bright or both dark it will pass between them. This being premised, the two plane faces under consideration belong to two planes given in space, the traces of which on the planes of projection can, consequently, be constructed, as well as the horizontal and vertical projections of their intersection; drawing, then, through any point whatever of this intersection a line parallel to the direction of the light, and constructing the two points in which it meets the planes of projection, if both these points lie without the traces of the proposed planes, the ray of light does not fall between the planes, and one is bright and the other dark; but if one or both of the points lie within the traces, it shows that the ray of light falls between the two planes, and that these planes

are either both bright or both dark: in the former case their intersection is a limiting edge; in the second it is not. Thus it can be determined beforehand, which are the edges to be dealt with in order to obtain the outline of the shadow of the proposed body.

The bodies treated of in the arts frequently presenting vertical edges, render the following observation of some utility. The horizontal projection of the vertical line is a single point; the line drawn through this point in the horizontal plane of projection, and directed towards the luminous point, always contains the horizontal projection of the shadow of the vertical on whatever surface this shadow be received. This is equally the case whether the luminous point be at a finite or infinite distance. In fact, in either case, the assemblage of rays passing through the vertical, forms a vertical plane, which must contain the shadow of the proposed vertical, and which will give it by its intersection with the surface which is to receive the shadow. The trace of this vertical plane on the horizontal plane of projection will, consequently, contain the horizontal projection of the shadow, whatever be the surface which receives it.

Moreover, this observation applies equally to every straight line perpendicular to any plane of projection whatever. The plane formed by the rays of light which pass through this line is also perpendicular to the plane of projection, and its trace on this plane must evidently contain the projection on this plane of the shadow cast by the straight line on any surface whatever. It thence appears that, in some circumstances, and by selecting with judgment the planes of projection, the above result will greatly simplify the work.

For the application of the preceding principles to particular examples recourse must be had to the lessons of descriptive geometry.

92. The case will now be considered, in which the body casting the shadow is not bounded by plane surfaces. The line which divides on the surface of the body the bright from the

dark part is no longer, in general, an assemblage of edges easily recognised; it is a curve which is to be determined solely by the property of being the boundary of these two parts. The rays of light received by the bright part, would penetrate into the body if they were produced: the dark part receives no rays, because to reach it, they would have to traverse the body which casts the shadow; but it is easy to see that the rays which proceed to the limiting curve of the dark and bright parts only touch the surface of the body. Each of these last rays, then, lies in a plane which touches this surface and passes through the luminous point. The curve under consideration can, therefore, be constructed by drawing through the luminous point a series of tangent planes to the surface of the proposed body, and by determining the points of contact; each of these points will belong to the required curve. The following mode of solution, however, is equally general, and more easily practised, from the nature of the investigations involved in it.

The luminous point is supposed to be at an infinite distance, and the direction of the rays of light to be indicated by the horizontal and vertical projections of a given line, to which these rays are to be parallel. The body which casts the shadow being known in form and position with respect to the planes of projection, as well as the surface on which the shadow is to be received, it is required to construct the projection of this shadow, and, with this view, to construct, on the surface of the body which casts the shadow, the curve dividing the dark from the bright part. This last investigation, besides entering into the solution of the present problem, is again interesting for the arts of design and of painting, since it shows on the surface of the illumined body, where the bright tints should terminate, and the dark tints commence.

The method about to be discussed, is analagous to that which has been given for the intersections of cylindrical surfaces.

Conceive a system of planes parallel to the direction of the light, and, moreover, perpendicular to one of the planes of pro-

jection, to the vertical plane for instance. The operations explained below for one plane of this system, will be easily repeated for the others.

It is to be remarked, in the first place, that, since it is perpendicular to the vertical plane of projection, the entire plane, as well as every line contained in it, is projected upon its trace. It can be conceived as made up of lines parallel to the direction of the light, or, what amounts to the same thing, of luminous rays. Now it will, in general, cut the surface of the body which casts the shadow in some curve. Of the rays of light situated in the plane, some will meet the curve and be stopped there; they evidently form part of the rays which are intercepted by the proposed body, and the interruption of which produces the shadow behind this body; others will pass outside the curve, and, meeting with no obstacle, will travel far into space; lastly, there will be some rays of light, which, situated between those which meet the curve and those which pass outside it, will merely touch it; and it is easily seen that, unless the body which casts the shadow be of infinite dimensions, there will, in general, be two such rays. These, being tangents to the section of the body made by the plane under consideration, are also tangents to the surface of the body; their points of contact then belong to the bounding curve of the bright and dark portions of the surface of the body; and the points in which they meet the surface on which the shadow is received, belong equally to the outline of the shadow.

It is these rays, then, that we must look for and construct; the property which characterises them will furnish the means. Since they are tangents to the curve of intersection of the plane under consideration with the surface of the body casting the shadow, their horizontal projections must be tangents to the projection of this same curve. The surface of the body is known, the cutting plane is given in position; if, then, the horizontal projection of their intersection be constructed, and if tangents to this projection be drawn parallel to the direction of the light projected upon the horizontal plane, they will be the projections of

the required rays, and the points of contact will be the horizontal projections of the points in which these rays of light touch the surface of the given body. The projection, or the trace of the cutting plane upon the vertical plane, contains the vertical projections of the rays of light; and to determine on these proportions, those of the above-mentioned points of contact, it is sufficient to draw through the horizontal projections of these points perpendiculars to the common intersection of the two planes of projection. There are thus obtained, by their horizontal and vertical projections, two points of the curve which on the given body divides the illumined portion from the unillumined.

If the operation just explained be repeated for any number of planes whatever, parallel to the light, and perpendicular to the vertical plane of projection, a corresponding system of points will be found on the horizontal projection, the curve passing through all which, will be the projection of the bounding curve, which, upon the given body, divides the illumined from the unillumined part. At the same time, another series of points will be found on the vertical projection, and the curve passing through them will be the vertical projection of the same bounding curve.

The determination of the outline of the shadow upon the receiving surface is still to be considered. The plane parallel to the light, treated of above, determines, in general, as has been seen, two luminous rays touching the surface of the body which casts the shadow, and which themselves lie in that plane. The points in which these rays meet the surface receiving the shadow belong to the outline sought. These points must evidently be situated on the curve of intersection of the plane with this same surface. The plane and the surface being known and determinate in position, the horizontal projection of their intersection can be constructed. Let this projection be constructed; the horizontal projections of the two rays of light under consideration will meet it in points which will be the projections of those in

which the rays themselves meet the surface; and these latter points belong, as has been stated, to the required outline. If, from the points thus obtained on the horizontal projection, perpendiculars be let fall upon the common section of the planes of projection, these perpendiculars will determine, by their intersection with the vertical projection of the cutting plane operated upon, the vertical projections of the same points of the outline of the shadow cast.

Repeating the last operation for each of the planes parallel to the direction of the light, a series of points will be obtained upon either projection, the curves passing through which will be the horizontal and vertical projections of the outline of the shadow of the given body upon the surface destined to receive it.

Amongst the planes parallel to the direction of the light, there may happen to be some which, after having cut the body casting the shadow, do not meet the receiving surface; or some of the rays touching the surface of the body, and determined by these planes, may not afterwards meet the curve of intersection of these same planes with the surface on which the shadow is supposed to be cast. In either case, these circumstances show that this surface does not receive the entire shadow cast by the body, but that a part escapes it, to be received by some more distant surface, or lost in space.

To facilitate the comprehension of the preceding, we proceed to apply it to an example.

Let a sphere be represented by the vertical and horizontal projections, A, A′, of two of its great circles (pl. X.,I fig. 39); and let the direction of the rays of light be given by the projections L L, L′ L′, of a line to which they are to be parallel; to find the horizontal and vertical projections, of the line which divides the bright portion of the surface of the sphere from the dark portion, and those of the outline of the shadow cast by the sphere upon a right cylinder with a circular base, given by its horizontal projection the circle B′.

Conformably to the method explained above, conceive a series

of planes parallel to the direction of the light, perpendicular to the vertical plane of projection, and, consequently, projected on this plane along their traces, $P\,p$, $P_1\,p_1$, $P_2\,p_2$, &c. Consider the plane $P\,p$; it will cut the sphere in a curve, whose vertical projection cannot but be on the trace $P\,p$, and whose horizontal projection will be the curve $p'p'p'p'$. After having constructed it, draw its two tangents $\Theta'\,\theta'$ and $T'\,t'$, parallel to $L'\,L'$, which will be the horizontal projections of two rays of light touching the sphere; while the vertical projections of these same rays can only be the trace $P\,p$ itself. The points of contact, T' and Θ', are the projections of the two points in which these rays touch the sphere, and, consequently, belong to the curve which separates, on its surface, the bright portion from the dark. To obtain the vertical projections of these same points, draw the two perpendiculars, $T'\,T$ and $\Theta'\,\Theta$, to the common section of the two planes of projection, and, producing them to meet the trace $P\,p$, the vertical projections, T and Θ, of the two points under consideration will be obtained. Repeating for each of the planes P_1, P_2, P_3, &c., the operation just explained for the plane P, there will be found on the horizontal plane the curve $T'\,T_1'$, $T_2'\,\Theta'$, $\Theta'\,\Theta_3'\,\Theta_4'\,T'_3$, and on the vertical plane the curve $T\,T_1$, $T_2\,\Theta$, $\Theta\,\Theta_3\,\Theta_4\,T_3$, for the projections of that which, on the sphere, divides the bright part from the dark.

Returning to the rays of light of which $T'\,t'$ and $\Theta'\,\theta'$ are the horizontal projections, and of which $P\,p$ is the vertical projection, and seeking the points in which they meet the surface of the cylinder; these will be points of the outline of the shadow cast on this surface by the sphere. The plane P cuts the surface of the cylinder in a curve, whose horizontal projection coincides with the circle which is the base of the cylinder. The lines $T'\,t'$ and $\Theta'\,\theta'$ meet this circle in the points r' and ρ', which are, consequently, the horizontal projections of the required points of meeting; to obtain their vertical projections, it is only necessary to draw the lines $r'\,r$ and $\rho'\,\rho$ perpendicular to the common section of the two planes of projection, and produce them to meet

the line $\mathrm{P}p$. This last operation being also repeated for the other planes, P_1, P_2, &c., there will be obtained the vertical projections of sundry other points of the outline of the shadow cast by the sphere on the cylinder, and the curve $r\,r_1\;r_2\,\rho_1\;\rho\,\rho_3\;\rho_4$, drawn through those points, will be the vertical projection of this outline.

On examining the plane P_3, and the two lines $\mathrm{T}_3'\,t_3'$ and $\Theta_3'\,\theta_3'$, which are the horizontal projections of the two rays of light touching the sphere in this plane, it will be observed that one of these projections, that which is designated by $\mathrm{T}_3'\,t_3'$, does not meet the base of the cylinder, which is the horizontal projection, as has been observed, of the section of the cylindrical surface by the plane P_3; the ray of light to which the projection $\mathrm{T}_3'\,t_3'$ belongs, does not meet, then, this surface, but passes on one side. It is thence to be concluded that the shadow cast by the sphere is not all received by the cylinder, and that the outline of this shadow on the cylinder surface does not form a complete enclosure, bnt terminates at the points where the rays of light touching the sphere are also tangents to the cylinder.

93. Up to the present point the luminous point has been supposed infinitely distant, and this hypothesis is that which is the most frequently employed, because it agrees very nearly with the manner in which bodies are illumined by the sun; but if the luminous point be supposed to be situated at a finite distance, the preceding method will be rendered applicable to this case by substituting for the parallel planes employed above, a set of planes passing through the luminous point, and, as in the former hypothesis, perpendicular to the vertical plane of projection.

The process above explained can often be simplified in particular questions, according to the generation of the surface of the body casting the shadow, and of that receiving it. For this purpose recourse must be had to the methods of descriptive geometry, of which many interesting applications may be made to these investigations. It is sufficient to have explained a mode of solution which comprises, in all its generality, the problem of

the graphical determination of shadows, when the luminous body is a single bright point.

The preceding remarks being confined to that part of the theory of shadows which has for its object the geometrical determination of their outlines, it remains to consider that which relates to the investigation of the intensity of the tints that must be given to the different parts of the shaded surfaces, to give them in the drawings all the appearances of shade and light, that the objects imitated present in nature; but to embrace this subject in all its extent, it is not sufficient to consider only, as hitherto, a luminous body, an opaque body, and a surface which receives the shadow, abstracting every accessory circumstanec; the objects must be studied with everything surrounding them, and regard must be had, among other things, to the position of the spectator and to the modifications undergone by the light before arriving at his eye, and producing there, the sensation of the spectacle upon which it fixes its view. These considerations appear to require that our remarks upon this matter should be preceded by the investigation of the Theory of Perspective.

THEORY OF PERSPECTIVE.

METHODS OF PLACING OBJECTS IN PERSPECTIVE, (pl. XI., fig. 40).

94. THE art of perspective consists in representing, on a picture of given form and in a given position, objects also given in form and position, as they would appear to an eye placed in a certain position. To comprehend this definition more distinctly, suppose the picture to be at first a transparent glass; if from all the given objects rays be conceived to proceed towards the eye, and in passing through the transparent picture, to leave upon it their traces imprinted with the colours and the tints belonging to the

points from which they set out, these traces will collectively form on the glass the complete representation of the objects: it is this representation which is proposed to be obtained in the art of perspective. It is seen that this subject, like the theory of shadows, divides into two distinct parts: the one, purely geometrical, has for its object the exact determination of the position on the picture of each point represented; the other, the investigation of the tint of shade and light that ought to be given to each part of the picture, which must generally be determined by physical considerations. This latter part, called *aerial perspective,* will be included in the investigations to be presently entered upon, to complete the theory of shadows; and the first part, called *linear perspective,* will now be discussed.

According to the definitions given above, it is easily perceived that *linear perspective* reduces itself to the construction of the section of a pyramid, the vertex and base of which are given, made by a determinate surface. The eye is the vertex; the base can be considered as spread over the surface of the objects to be placed in perspective, and the cutting surface is the picture.

The methods of descriptive geometry easily give the solution of this problem taken in all its generality, that is supposing the picture to be any curved surface whatever. However, keeping especially in view what is of constant utility in the arts, that only which concerns perspectives drawn on plane surfaces will be discussed with some detail, and a few observations will then be added respecting perspectives constructed upon curved surfaces.

The picture will be supposed to lie in a vertical plane, or perpendicular to what is considered the horizontal plane of projection; it could without difficulty be supposed to be inclined to these planes in any manner whatever; but the hypothesis proposed is more natural, and makes the constructions more simple.

Thus, the position of the eye, that of an object of known form, and, lastly, that of a vertical plane, being given with re-

spect to the planes of projection, it is required to find the intersections of this plane with the straight lines drawn from the eye to each point of the proposed object, and to transfer them to a picture representing this same vertical plane supposed to be thrown down.

The points of intersection will be given with more or less advantage and facility by different constructions, according to the respective positions of the object, the eye, and the picture; that which is the most simple and usually the most advantageous will be first explained.

Let the vertical plane of projection be placed in such a position that the plane of the picture may be perpendicular to it, and may, consequently, be projected upon it in a vertical line, which will be its trace. Let O′ and O″ (pl. XI., fig. 40) be the projections of the eye, T′ T′ and T″ T″ those of the picture, or the traces of the vertical plane to which it belongs; suppose, moreover, that the projections of the objects to be placed in perspective are already constructed, or that we begin by constructing them on the planes of projection adopted; for example, let the object be a pyramid on a quadrangular base, of which the vertices or solid angles, A, B, C, D, E, are given in horizontal projection at the points A′, B′, C′, D′, E′, and in vertical projection, at the points A″, B″, C″, D″, E″.

If a line be drawn from the eye to the point A of the proposed object, the projections of this line will be the straight lines O′ A′ and O″ A″. The points *a*′ and *a*″, where these straight lines cut the projections T′ T′ and T″ T″, of the picture, are evidently the projections of the point in which the visual ray meets the picture; and it only remains to find the position of this point on the picture itself, which we shall conceive to be removed from its position T′ T′ T″ T″, and placed at M N. A simple means of accomplishing this is to choose on this picture two lines to be taken for axes, to which to refer all the other points; the position of these axes being fixed on the planes of projection, the distance of each of them must be sought from the intersection

of the visual ray with the picture, and by means of these distances it will be easy to mark the place of the point upon the picture. As these two axes can be chosen arbitrarily, we shall suppose two planes to be drawn through the eye, the one horizontal and the other vertical, both perpendicular to the picture; their traces on the planes of projection will be $O'Y$ and $O'X$; they will cut the plane of the picture in two lines, one horizontal, represented in vertical projection by the point x, and the other vertical, represented in horizontal projection by the point y; these two lines will be the axes adopted, and will be represented on the picture by XX and YY.

This being premised, since a' is, as has been stated, the horizontal projection of the point in which the visual ray drawn to the point A meets the picture, $y\,a'$ will be the distance at which this point must be placed from the vertical passing through the point y, or from the axis YY on the picture MN. If, then, on this picture, a parallel to the axis YY be drawn at a distance equal to $y\,a'$ to the right or left of this axis, accordingly as in the horizontal projection a' is to to the right or left of y, this parallel $a\,a'$ will contain the required point. Moreover, a'' being the vertical projection of the same point, $x\,a''$ measures the distance of this point from the horizontal axis passing through the point x in the picture; drawing, then, on the picture a'' a parallel to the axis XX, above or below this axis, accordingly as in the vertical projection the point a'' is above or below the point x; the intersection of the two lines $a''\,a$, $a'\,a$, parallel to the axes, will be the required point, or the perspective of the point A. The same operation can be gone through for every point of the pyramid, by which means the complete perspective will be obtained.

The work, however, will be much abridged by a few observations. It will be remarked, in the first place, that, when the picture is a plane surface, the perspective of a straight line is a straight line. In fact, the visual rays drawn from the eye to the several points of the proposed straight line are, in the plane

drawn through this line and the eye; consequently, the points in which they meet the picture must be on the straight line, which is the intersection of this plane with the picture. Thus, it is sufficient to construct the perspectives of two points of the proposed line, and join them by a straight line, to obtain the perspective of the line itself. In the example given above, it is only necessary to construct the perspectives of the five vertices, A, B, C, D, E, of the pyramid; and the straight lines joining them will be the perspectives of the edges.

In the second place, if the body to be placed in perspective be opaque and impenetrable to the visual rays, the anterior part will intercept the view of the other part; it is useless, therefore, to construct the perspective of the points belonging to the latter: thus, in the given example, the point E of the pyramid not being visible to the eye placed at O, it is useless to find the point corresponding to it on the picture M N.

The visible portion of an object is separated from that which is hid from view, by a line called the *apparent outline.* The perspective of the apparent outline is the sketch which, on the picture, envelopes the image of the object to be represented; it is, therefore, important, in general, to determine accurately the apparent outline of an object, and carefully to construct its perspective.

When the objects to be represented are bounded by plane surfaces and rectilinear edges, it is generally easy to distinguish the visible faces for any given position of the eye, from those which are hidden, and, consequently, to recognize those of the edges which collectively form the apparent outline. But, when the objects are bounded by curved surfaces, the apparent outline is a curve on the surface of the body, which must be determined by means of its property of separating the portion of the body visible to an eye placed in a given position from that which is concealed. It is seen that this investigation is exactly the same as that of the line which, on an opaque body, divides the illumined from the dark part, when the luminous body is a single point,

placed at a finite distance. Instead, therefore, of repeating this investigation, it is only necessary to refer to the solutions already given of questions perfectly analogous in the Theory of Shadows.

95. A result of perspective of great importance from its frequent applications, and to which attendance is essential to correct drawing, must here be noticed; it consists in this, that whatever be the form of the picture, the perspectives of any number of straight lines parallel to one another, but not to the picture, meet in one point. If the picture is a plane, these perspectives are themselves straight lines, all passing through the same point, a proposition which is easily demonstrated.

In fact, a straight line being given to be placed in perspective, the assemblage of visual rays drawn from the eye to this line, forms a plane passing through the eye and the line, the intersection of which with the picture is the required perspective; a line, then, drawn through the eye, parallel to the given line, is contained entirely in this plane. If, now, there be a second line parallel to the first to be also placed in perspective, the intersection with the picture of a plane passing through this second line and the eye will be its perspective; and a line drawn through the eye parallel to the line will be contained entirely in this second plane. Now the two given lines being parallel, the line drawn through the eye, parallel to either of them, is the same, and, being at the same time in both planes, is their intersection: the point in which it meets the picture is, therefore, the point of intersection of the lines in which these planes cut the picture, or, what amounts to the same thing, that in which the perspectives meet. It follows from this, that to place any number of parallel lines in perspective, it is only necessary to draw through the eye a line parallel to them; and the point in which this last line meets the picture will be the point to which the perspectives of all these lines must be directed.

The projections of the line drawn through the eye are parallel to those of the line to be placed in perspective, and are, consequently, easily constructed; the traces of the picture on the

planes of projection are given: the point on which the line drawn through the eye meets the picture is, therefore, readily found.

The result just explained abridges the work very considerably, when the picture is a plane surface, and it is required to trace the perspectives of different parallel lines. In this case, these perspectives are themselves straight lines, and the point in which they all meet being determined, as above pointed out, it will be sufficient, in order to trace them, to find on the picture the perspective of a second point for each of them.

But it is not only as a means of abridgment, that the process just pointed out is of importance; it is, moreover, the safest method of avoiding errors, which are offensive to every eye. We generally take less notice of the actual magnitudes of objects, than of the parallelism of lines that we consider ought to be parallel. When two lines are a little closer, or a little more open than they should be, it requires a practised eye and some degree of attention to perceive the defect; but, if they are not parallel when they should be so, it is perceived at a glance, and offends immediately. If, then, when several parallel lines are placed in perspective, the perspectives are not all directed to the same point, this error shocks the observer, and the parallels do not appear to him to be so. Thus it is always essential to determine on the picture the point of concurrence of lines which represent the perspectives of parallel lines, in order to make sure that the perspectives pass through this point.

In the explanation of the process of construction given above, the vertical plane of projection has been supposed to be perpendicular to the plane of the picture, by which disposition the advantage has been gained of having the whole picture projected on a single line. If the picture were oblique to the vertical plane of projection, to find the height of each point of the perspective above the horizontal axis to which it is referred, a perpendicular must be let fall on the intersection of the two planes of projection, from the point in which the horizontal projection

of the visual ray meets the horizontal trace of the picture; and this perpendicular must be produced to meet the vertical projection of the visual ray. This work, although tedious, will, in some circumstances, be found less troublesome than the preliminary construction of a vertical projection on a plane perpendicular to the picture.

Suppose, for instance, a series of similar pilasters are to be put in perspective, and that their direction be oblique to the plane of the picture; it would take up too much time to project them on a vertical plane perpendicular to the picture; but forming their projection on a plane perpendicular to the direction of the pilasters, it will reduce to the projection of one of them. It is seen, then, to be preferable, in this case, to adopt this latter arrangement, notwithstanding the inconvenience of having to trace a line more to construct the perspective of each point.

96. In general, the problem presented by linear perspective, considered in an elementary point of view, reduces itself to the construction of the point in which the visual ray, drawn from the eye to a determinate point, meets the picture; and it is useful to be acquainted with various means of resolving it, in order to make use, in each particular case, of the least troublesome. The greater part of the methods given in works which treat of perspective, and particularly that developed above, are included in the general mode of solution which we proceed to discuss.

If, through the point to be placed in perspective and the eye, two different planes be conceived to be drawn, the visual ray will coincide with their intersection, and as they will necessarily cut the picture, if the lines or traces in which they cut it be drawn, the point in which these traces intersect will belong to the intersection of the two planes, and will, therefore, be the spot where the visual ray meets the picture. It is for the draughtsman to choose, amongst the infinite number of planes which can be drawn through the eye and the point to be placed in perspective, the two planes, the traces of which on the picture he can most easily determine. By taking them perpendicular to the planes

of projection, each to each, the method of construction given above is fallen into. It may often be advantageous to suppose one of the planes perpendicular to the picture itself; in which case, as is readily seen, its trace will pass through the feet of the perpendiculars let fall from the eye and from the given point upon the picture. Generally, if two parallel lines be conceived to be drawn through the point and through the eye, the intersection of the picture and the plane containing them will pass through the points in which these parallels meet the picture itself.

These various observations are sufficient to enable persons conversant with the methods of descriptive geometry, to abridge, in a great number of cases, and to simplify very much, the necessary operations in the practice of linear perspective.

Suppose, now, the picture to be no longer a plane, but a given curved surface; the considerations above discussed will generally lead, for each case, to the most advantageous construction. For, amongst all planes passing through the eye and the point of which the projection is required, and, consequently, containing the visual ray, that plane can always be selected which, according to the nature of the surface proposed for the picture, gives by its intersection with this picture, the easiest curve to construct, either on the plane itself, or in one of its projections. It will then be easy to find the intersection of this curve with the visual ray, which will determine the point where the ray meets the picture.

If, for example, the picture were a spherical surface, the plane drawn through the eye and the point to be placed in perspective should be made to pass through the centre of the sphere also; then the intersection would always be a great circle, the intersection of which with the visual ray would be easily found on the plane itself.

If the picture were a conical surface, the plane containing the visual ray should be made to pass through the vertex of the cone; the intersection of this plane with the picture would be a straight

line, the projections of which would be found without difficulty, and also their intersections with the projections of the visual ray.

Panoramas are perspectives drawn on vertical cylindrical surfaces, whose bases are circular, the point of view being taken on the axis of the cylinder. To place any point in perspective on the surface of a vertical cylinder, conceive a vertical plane, drawn through the eye and the proposed point, cutting the surface in a vertical line to be determined by the intersection of the horizontal trace of the plane with the circumference of the circular base of the cylinder. Construct the vertical projection of this line, and its intersection with the vertical projection of the visual ray will determine the height, above the base of the cylinder, at which the visual ray meets its surface; and it will now be easy to construct the perspective of the proposed point, either on the surface of the cylinder itself, or on its development.

97. The means of resolving all the questions that can occur in perspective having now been given, a few further remarks only will be added.

Having a picture which presents the perspective of an object, taken in a given point of view, the trace can be deduced of a perspective, on a different picture, of the same object taken from the same point of view. For, the eye and the first picture being determined in position, the direction of the visual rays drawn from the eye to each point of the object represented is fixed, and, consequently, the points can thence be found, in which they meet the surface of the other picture, the position of which is given.

If, however, the point of view were changed, the case would be completely altered, there being, then, nothing to determine the direction of the visual rays, and a single perspective not being sufficient to define the object represented. A perspective is a sort of projection differing from the orthogonal projection in common use, only in having its points projected by lines directed to the point of view whence the perspective is taken, instead of lines perpendicular to the plane of projection: now we know that an object is not completely determined without the assistance of

two projections; in like manner it can be completely determined only by the aid of two perspectives, for each of which the position of the point of view is known.

The investigation of the geometrical part of the Theory of Shadows will terminate here. The methods explained embrace nearly everything respecting the representation of objects, which, in practice, is susceptible of rigorous imitation. Thus, various objects being proposed and determined by their projections, if they are supposed to be illumined in a known manner, construct first the outlines of the bright and dark portions on the surface of each of them, and those of the shadows cast by them, one on another; then trace on a picture of given form the perspectives, taken from a given point, of these same objects, as well as of the outlines of their shadows: it will only remain, for their complete representation, to give to the different parts of their images the tints with which, in reality, they present themselves to our notice.

ON THE DETERMINATION OF TINTS FOR THE CORRECT REPRESENTATION OF OBJECTS, AND ON AERIAL PERSPECTIVE.

98. The part of the theory of shadows and of perspective now to be discussed is extremely complicated, and must be studied with great care; it requires some knowledge of physics, and especially the observation of a great number of facts.

Unfortunately, artists, who are compelled to be constantly reflecting on this subject, seldom publish the results of their meditations. Perhaps very many curious discoveries and important observations remain unknown and lost for the purpose of general instruction, because the artists who have made them have either been unable to render an exact account of them, or have neglected to do so. The following remarks are far from being presented as a complete exposition of the subject; they are but ideas thrown in advance, and intended to open a career almost novel, in the hope that these essays may give rise to more pro-

found researches, and thus become the foundation of some ulterior progress in the science.

The tint which an illumined object offers to our view, depends primarily upon the peculiar intensity of the light received from the luminous body, and returned to our eye, and upon the manner in which its distribution over the surface of the object, and the reflection which causes it to come to us, takes place; secondly, upon the modifications that the light undergoes from the effect of the medium or of the atmosphere that it passes through, and of the other circumstances to which it is submitted. In this order the following considerations will be arranged.

To begin by investigating the intensity of the light coming from luminous bodies to an illuminated object, suppose, for greater simplicity, the luminous bodies to be but one, and consider this as reduced to a single point. Since it is known that the intensity of the light emitted from a luminous point decreases in the inverse ratio of the square of the distance, it is evident that the further from the luminous point the illumined object lies, the less will it be illumined. This observation is not of any great importance in the art of design, because the objects are constantly supposed to be illumined by the sun. In this case, from the immense distance of the luminous body in proportion to the dimensions of the illumined objects, and to their distances apart, it can be regarded as the same for them all, and, consequently, to make no difference in the intensity of the light thrown on the various points of the objects under consideration; but to represent a nocturnal scene, illumined by a lamp or a fire, regard must be paid to the distances of the illumined objects from the luminous body, and greater brightness must be given to those that are to be made to appear nearer to the point whence the light proceeds.

The preceding remark relates only to the illumined portions; with respect to the portions in shade, when there is supposed to be but one luminous point, and everything that can reflect the light to be removed, they have all an equal intensity—they are all absolutely black. This assertion may perhaps appear extraordi-

nary, since we are not accustomed to see bodies illumined in this manner; the sun is the principal source of the light in the day; but the other bodies reflect it, and return it to us, so as to throw light where the direct rays of the sun do not arrive, and we never have the opportunity of observing a complete shadow; an idea of it can be formed only from experiments in a dark chamber, and particularly in that of the solar microscope. When a pencil of solar rays is introduced into the dark chamber, and made to fall on a convex lens, its rays converge to a focus, then cross, and become divergent, forming a cone of light, which is thrown, as a very bright circle, on the opposite wall of the chamber. If a perfectly white tablet be placed to receive this luminous circle, and an object be put in front of it which intercepts a portion of the rays, the shadow will appear of the most intense blackness, and will be bounded by a very precise and strongly marked outline. In this case, the light, in fact, proceeds from a single point, viz., the focus of the lens through which the luminous rays pass, and there is not sufficient reflected light to diminish sensibly the obscurity of the dark chamber in the parts unvisited by any direct rays.

99. The light returned by the illumined object to the eye of the observer is now to be considered. If it had to traverse a perfect vacuum, two objects of the same brightness would appear to the eye equally bright, whatever might be their distance from the observer. To render this clear conceive two equal circles, equally illumined, and situated on planes equally inclined to the rays drawn from their centers to the eye: the intensity of the light remitted by each of them to the eye will decrease in the inverse ratio of the squares of their distances from it; but the size of the pictures of these circles upon the retina of the eye will also decrease in the inverse ratio of the squares of the same distances. Thus, while on the one hand the light transmitted by every point of the more distant circle is less intense, on the other hand, it is more crowded together and condensed to form a more compact image; these two contrary effects, varying according to the same

ratio, counteract each other in producing the sensation of vision, and hence it follows that the two circles, placed at unequal distances, ought notwithstanding to appear equally bright.

However it is not so in nature, because the atmosphere which the light traverses is not completely transparent. But, before seeking to appreciate the effect of this imperfect transparency upon the luminous rays, it will be proper to examine the disposition of the light at the illumined surface, both with regard to its distribution there, and with regard to its remission to the eye.

The surfaces may be divided into two classes, according to the manner in which they receive and remit the light; viz., polished surfaces and dull surfaces.

No surfaces perfectly polished are to be met with, but those which form mirrors may be considered as approximating to this state. It is known that the rays of light which fall upon a polished surface are reflected so as to make the angle of reflection equal to the angle of incidence. If the light emanates from a single point, each point of the polished surface receives and reflects but a single ray, and of these rays one only arrives at the eye, while the others pass on one side of it: the eye, then, perceives only the point of the surface which remits this ray, the remainder is completely invisible to it, and the visible point appears the brighter in consequence. The surface, the position of the eye, and that of the luminous point being known, the determination of the brilliant point is a problem of descriptive geometry, the solution of which is more or less complicated, according to the nature of the given surface: it is required, in fact, to find a point on this surface, such that, drawing from it straight lines to the eye and to the luminous point, these lines shall be in a plane perpendicular to the tangent plane, and make equal angles with it. It is easy to see, that, supposing the polished surface of sufficient extent, it must, generally, have a brilliant point, and a brilliant *point* only as long as the light emanates from a single point. If it comes from a body of finite dimensions, se-

veral points of the polished surface remit to the eye rays which collectively present to it the image, more or less distorted, of the luminous body; the remainder of the surface has a blackness the more intense the higher the polish of this surface. When, then, a polished body is to be represented, after having determined the position of the brilliant point, this point must be painted a very bright white, and the remainder of the body must be kept in shade.

Dull surfaces, which form the second class, a much larger class than the first, differ from polished surfaces in remitting rays to the eye from all their points at which rays arrive from the luminous body, unless an interposed body obstruct them.

It is easy enough to form a precise idea of the quantity of light that each part of any surface whatever receives from a luminous point. Abstracting the obliquity at which the surface presents each of its parts to the light, the intensity of the light which reaches it is inversely as the square of its distance from the luminous point. Moreover, if this point be supposed the center of a sphere, the quantity of rays received by an element of the illumined surface will be measured by the portion of the surface of the sphere comprised within the cone, the vertex of which is the luminous point, and its base the element of the given surface. The more oblique this element be to the rays received by it, the more contracted will be the cone, and the less the portion of the surface of the sphere comprised within it. Hence it appears that the more obliquely the illumined surface presents itself to the luminous rays, the less light it receives. These results are expressed in mathematical terms by saying, that, *for each point of the surface, the intensity of the light is directly proportional to the sine of the angle made by the incident ray with the tangent plane at this point, and inversely proportional to the square of the distance from the luminous point.*

It is more difficult to appreciate satisfactorily how the light is reflected by dull bodies, and what amount each part of their surface transmits to the eye. This investigation depends upon the texture of the face of the bodies, and our physical knowledge

is, in this respect, too scanty to furnish us with the necessary data. The following remarks will be, therefore, based upon certain hypotheses, the results will only be probable, and are put forth only until they can be replaced by others based upon a more certain theory.

Let it be allowed, then, that each of the molecules belonging to a dull surface acts after the manner of a luminous point in reflecting throughout the whole of free space the light received by it and not absorbed. It is perceived that these molecules must present an infinite number of faces, which are only made sensible to us by the dull appearance of the body, and which do not prevent its surface from appearing uniform and continuous. On this hypothesis each molecule, situated on the surface of the body, remits to the eye a ray of light. Considering an element of the surface, it has been seen that the distance of this element from the observer influences the magnitude of the image presented to his eye, but not its brightness, if the effect upon the light of the defective transparency of the atmosphere be disregarded. The whole of the rays reflected to the eye by all the points belonging to this element form a cone of which the element is the base, and the eye the vertex; and the number of rays comprised in this cone is proportional to the extent of the element of the surface. If a sphere be conceived, having the eye for its center, and a radius equal to the distance between the eye and the base of the cone, the portion of the surface of this sphere comprised within the cone, will be the measure of the angular space in which the rays are contained. The intensity of the light arriving at the eye may, therefore, be estimated by the ratio of the extent of the element under consideration, to that of this portion of the spherical surface. The extent of the element remaining the same, that of the corresponding portion of the spherical surface will be the smaller, the more acute the angle made by the element with the visual rays; so that, the intensity of the light reflected by the dull surface will be less the more nearly perpendicular the surface be to the rays remitted to the eye; and this is

expressed in mathematical terms by saying, that, *for each element of the surface, this intensity is inversely proportional to the sine of the angle made with the tangent plane by the visual ray.*

This result must not be interpreted too strictly when the angle spoken off is very small; in this case, the protuberances of the dull surface, partly covering one another, deprive us of a portion of the light that would otherwise reach us. Thus, in looking at a plane dull surface under a very acute angle, it does not appear of an intense brightness, as indicated by the proposed analytical expression: this expression then fails, from not taking account of the little asperities with which the surface is covered, and of the ratios of their dimensions to the distances between them.

The following application of the above result is remarkable.

The moon may be regarded as a dull body illumined by the sun, whose rays it remits to us. If this satellite were enveloped with an atmosphere, the rays remitted to us from the extremities of its disk would have to traverse a greater breadth of this atmosphere, and would no doubt arrive here more feeble than those coming from the center. But astronomical observations show that the moon has no sensible atmosphere; and, in consequence of its spherical form, a greater extent of surface would be seen near its extremities under the same visual angle; more reflected rays ought then to reach us from these parts, and the extremities ought consequently to appear brighter; and, in fact, the brightness of the moon is observed to be more intense round the edge of its disk than in its center.

Nature presents to our view a great number of bodies, the surfaces of which are intermediate to the two extreme classes considered above, and participate to a certain extent, as experience shows, in the properties of both polished and dull surfaces. With regard to these bodies, the molecules appertaining to their exterior envelope may be supposed to be small spheres, nearly polished, reflecting the light in part, after the manner of polished bodies, and more or less embedded in the body itself,

as its polish is more or less perfect. If they were isolated, each would exhibit a brilliant point; but as a part only of their outline is visible, all of them cannot present a point of this kind: this property is possessed by those alone, for which the brilliant point falls upon their anterior and visible segment, which blends to our view with the general surface of the body. Hence it may be concluded that the point on the surface of the body, which would be the brilliant point in the case of a polished surface, will be, in some sort, the center of the part of the surface containing the polished molecules capable of exhibiting to us brilliant points; and it is perceived that this luminous portion will be less contracted the less smooth the body be; or, in other words, it may be said that, for imperfectly polished bodies, the brilliant point expands and spreads with a diminished lustre over a larger space, as the polish is less perfect. Throughout the remainder of the surface of the proposed body, the molecules only remit the light after the manner of completely dull surfaces, and the previous remarks upon this case again apply.

Up to this point the following circumstances only with regard to the light have been considered, the intensity with which it arrives at the body, is distributed there, and remitted on its passage to the eye of the spectator; no account has been taken of the charges produced by the media which it traverses, or by the other influences to which it may be subjected: the modifications due to these causes have still to be studied.

100. The air traversed by the light not being perfectly transparent, its molecules arrest some of the rays of light and reflect them, as opaque bodies do. The effect of this is imperceptible with near objects, but upon distant objects it is striking; it extends over the illumined parts as well as the parts in shade; it diminishes the intensity of the brightness of the former, and of the shade of the latter, and modifies the colour of the objects.

The light reflected by the molecules of the air has a determinate colour, the atmosphere, like every other body in nature, having its peculiar colour; it is this which forms the blue of that

which is called the sky. If the atmosphere had no existence, or did not remit light, the sky would appear of an absolute blackness, upon which the stars would form brilliant points. The blue of the sky is the more vivid the less the humidity in the atmosphere; and it is for this reason that the sky of southern climes is always of a darker azure than that of the north.

When, then, a pencil of light traverses the air to a sufficient extent, it loses on the way a portion of the rays of which it is formed, and, consequently, of its intensity.

This observation is not so important in considering the ray of light in its progress from the luminous body to the illumined object, as in following it on its return, as the visual ray, from the illumined object to the eye. In fact, as regards every object illumined by the sun at the same instant, the light traverses a layer of atmosphere of sensibly equal thickness to reach each of these, and the loss that it suffers in its progress diminishes equally the brightness of them all. There are, however, circumstances in which it is essential to consider this loss; and, to represent in a picture the effect of the rising sun, a painter observes that the light, traversing horizontally a great breadth of atmosphere before arriving at the objects coloured by it, has less power and brightness than in the middle of the day.

But it is especially in the passage of the light from the illumined object to the eye, that it is essential to examine how the light is affected by the mass of interposed air. Not only is a part of the rays reflected by the object, interrupted, but the intermediate molecules of air receive also direct rays of light, and reflect them with their proper colour in the same direction with those remitted to the eye by the illumined object. The sensation to be produced upon the eye by this object is, therefore, affected in two ways; first, by having a portion of the rays, going to produce it, arrested; and, secondly, by the mixture with these of foreign rays of a bluish colour. This effect is the more decided, the more considerable the interposed mass of air; and it may be admitted as a principle that, as the distance of the illumined

objects from the eye increases, their brightness diminishes, and their peculiar colour partakes more of the blue colour of the atmosphere.

Upon objects in the shade an analogous effect takes place. If there were only a single luminous point and no atmosphere, the shade would be absolutely black; but the surrounding objects, and particularly the air itself, illumine to a certain extent the parts of the bodies which receive no direct light, and it is thus that their forms become perceptible. Moreover, the rays remitted by them to us are also in part arrested by the molecules of the intermediate air; and these molecules receive and reflect to the eye other rays, which, coming to the eye from the direction in which the shade is situated with respect to the spectator, diminish the intensity of this shade by mingling with it a bluish tint; it may, therefore, be also admitted that, the more distant the unillumined objects, the more the shade diminishes in intensity by approximating to the tint of the atmosphere.

Conceive two rows of similar objects, extending to a considerable distance; the one composed of illumined objects, and the other of objects in shade. The brightness of the objects which compose the first will go on decreasing as they are further removed; if they are supposed to be white, the white will diminish in brilliancy, and will, moreover, change in colour, by insensible degrees from one object to the next, but in a marked manner on the whole length of the row, and will pass into a bluish tint. At the same time the shade of the objects which compose the second row will diminish in intensity; they will brighten up, not by approximating to a white colour, but to a blue. If the two rows of objects under consideration were produced to an extreme distance, it would at length happen that the white of those which are illumined, and the black of those which are in shade, decreasing continually by approximating to blue, will be lost by blending with the colour of the atmosphere. This is observed in viewing high mountains at a distance of twenty-five or thirty leagues; their tops crowned with snow and of a dazzling brilliancy, their

grand shadows, so decided, when viewed from a short distance, and on a fine day, are almost wholly lost, and melt away in the azure of the sky.

Thus to produce in a picture the sensation of the different distances of two objects, the more distant is to be painted of less vivid colours, by abating the lights and diminishing the intensity of the shadows; and when the objects to be represented are very distant, the colours must take a general bluish tint.

This principle is well known, and is often exaggerated by an abuse of it which it will be useful to note. As stated above, it is only when the differences between the distances of different objects from the eye are considerable, that there results a perceptible difference in the effects produced by the interposed masses of air upon the light remitted to us by the objects. In looking, for example, at an architectural facade, a portion of which forms a fore-building, projecting about a yard, the layer of air a yard thick, which the rays coming from the further part of the facade have to traverse, more than the others, to reach the eye, will take away none of their intensity, or at least will take away too little for the diminution to be appreciable to our senses. Supposing, then, the fore-building and the back portion of the front to be parallel and similarly illumined, it is wrong to make a difference in the tints given them, as is the custom of very many draughtsmen: they would appear to us equally illumined, and should be represented with the same brightness.

We perfectly distinguish, however, in the reality, that one part projects before the other; it is not necessary for this purpose that the fore-building should cast a shadow on the further part; and even when the direction of the ray of light from the sun, and the position of the eye, are such that no shadow is apparent, we judge without difficulty which is the nearest plane, and which the most distant. It is essential to discover the circumstances which lead us to this judgment, in order to imitate them if possible, so that the picture may inform the eye of the relative positions of the planes in the same manner as the reality.

Let us still suppose before us an architectural facade, of a perfectly uniform tone of colour, and of which one part projects before the other. If any obstacle, as a plank, be placed so as to cover the view of the edge by which the fore-building is bounded, it will become impossible to tell which of the two parts is the nearest to the eye; but if the obstacle be removed, we can judge in an instant. This simple experiment teaches us then that it is the manner in which the light acts upon the bounding edge of the fore-building which apprizes the eye of the existence of a projection. If this edge were a mathematical straight line, the action of the light upon it would be perfectly inappreciable, and there would still be no means of distinguishing which was the projecting part. But this edge is never sharp, never a mathematical straight line: the materials of which it is constructed are not absolutely free from porosity, the tools used in shaping them are not perfect; an infinite amount of precaution is not taken in cutting them, and on coming from the hands of the workman, their edge was far from being strictly perfect. Since then, everything striking, or merely rubbing, against it, has blunted it more; and really, instead of being a sharp edge, it is nothing but a rounded surface that can be considered as a portion of a vertical circular cylinder of a very small radius: it is by the manner in which the light acts upon this cylindrical surface, and is thence remitted to the eye, that the existence of the projection is indicated.

It has been already shown that each part of a curved surface receives more light the more directly the luminous rays fall upon it, and that the light remitted by it to the eye has more intensity the more obliquely the surface presents itself to our view. According to these principles, a portion of the small cylindrical surface representing the edge on the side which faces the light, must be made of a brightness more vivid, and a portion of the other edge of a brightness less than that of the facade of the building, the whole depending, for its precise determination, upon the position of the eye, and the direction of the rays of light.

Thus to produce the sensation, in the proposed example, that

a part of the facade projects, there must be placed upon the edges, on the side of the shade, a line a little less bright, and on those on the side of the light, a line more illumined, which is called the reflection; for the rest, the tint of the two parallel planes composing the facade must be the same.

Some explanation still require to be added which belong to other considerations.

Our organs are endowed with certain properties which alter the sensations that they convey to us. The organ of vision, for instance, prolongs the sensation beyond the instant at which it is excited, as is shown by a well-known experiment; when a piece of glowing coal placed at the end of a stick is moved with rapidity, the appearance is produced, not of the coal moving successively from point to point, but of a continuous band of fire.

This organ also possesses the property of extending and magnifying objects more the more highly they are illumined; take the following striking example: some days after the new moon, and when she approaches her first quarter, a little after sunset she is still visible above the horizon; about a quarter only of her disk is illumined, but the remainder receives by reflexion some light from the earth, and is consequently visible; now, the illumined part appears to have a much larger diameter than the part in shade, and there seems to be a considerable jump from the curvature of the one to that of the other. At the time of her last quarter the same illusion is repeated before sunrise; but the part in shade during the first quarter is then the illumined part, and appears in its turn larger than the other, which has become dark. Many experiments confirm this faculty of the organ of vision to increase the dimensions of white and illumined objects at the expense of those which are dark; the following is one of the most simple; place side by side several parallel bands, of perfectly equal breadth, and alternately black and white, and, looking at them from a point at a little distance, the white bands will appear much broader than the black.

A third property belonging to the eye in common with our

other organs consists in this, that in general strong sensations weaken momentarily the perception of sensations less powerful. It is thus that the gunner who comes from hearing the discharge of a battery is insensible to the impression of a moderate noise. It will even happen that a vivid sensation experienced by one organ will alter altogether a sensation afterwards received by a less sensitive organ. Before tasting a liquor, we perceive its perfume; but, as soon as we have drank some drops, our smell becomes insensible to it; the strong sensation experienced by the palate blunts entirely the sensibility of the smell. This effect of vivid sensations is very remarkable upon the organ of vision; brilliant objects render us insensible to those which have a less amount of light; upon passing from broad day into a place but dimly lighted, nothing can at the first moment be distinguished, the persons nearest to us can with difficulty be recognized; but little by little the vision becomes accustomed to this feeble light, and, after some time, one can even read small print. It is true that at the moment of passing from the light to the comparative darkness, the pupil of the eye becomes dilated and permits a greater number of rays to enter; but this dilatation of the pupil takes place instantaneously, and is not the cause of the effect above recited; this arises from the eye parting but slowly with the impression left upon it by the brightness of the broad daylight.

In applying these remarks to the determination of the reflection that ought to be given to an illumined edge, it will be perceived that this reflection would appear to the eye a little broader than it is in fact, and that the contiguous parts would appear a little darker. In order to reproduce in the picture these appearances, essential to the accuracy of the representation, a greater breadth must be given to the reflection, and, parallel to it, for a little distance on both sides, must be placed a more sombre tint. If we had at our disposal colours as vivid as those of nature, if we could paint the reflection of a white as brilliant, as the reality, it would be vain to give it greater breadth, and to enhance it, in some sort, by the contrast of more sombre tints

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placed alongside: the faithful copy of nature would reproduce on our organs the effect produced by the object itself; but we are obliged to compensate by a species of exaggeration, easily employed, for the imperfection of our means of imitation.

ON THE CHANGES UNDERGONE BY COLOURS IN CERTAIN CIRCUMSTANCES.

101. After having treated of the modifications experienced by light in its absolute intensity, and of the colours of which it excites the sensation, it remains to examine what changes are produced in the colours themselves by the action of different modifying causes. This research belongs to the branch of optics which comprises the study of coloured light; it is much too vast to be here embraced in its entirety, and we shall confine ourselves to a few observations which we believe to be susceptible of frequent application.

One of the principal causes of the variations of the colours consists in the nature of the luminous body; thus the cornflower of the fields, which is of a beautiful blue during the day, appears violet by candle-light; by the same light, the green of leaves and plants becomes much more sombre, and yellow approaches very nearly to a white tinged with pink: this is the reason why persons of not very fair complexions appear to more advantage by this light.

But the changes observed in colours do not proceed only from the nature of the light, either direct or reflected, with which they are illumined; they frequently arise, in part, from an imperfect appreciation that we form of colours when our judgment is, so to speak, biassed by particular circumstances: witness the following examples.

In the morning before sunrise, and when the sky is of a beautiful azure, if on a table before an open window a sheet of white paper and a candle be placed, the paper will be illumined at the same time by the flame of the candle, and by the light spread through the atmosphere, and remitted by the air to us.

In these circumstances, placing a body to intercept a portion of the light thrown by the candle on the paper, the shadow cast upon the paper will be illumined only by the atmosphere; it will appear of a beautiful blue, as it should, because the light reflected by the atmosphere is blue; but, if the candle be extinguished, the whole of the paper will be illumined only by this same blue light, and yet we shall not hesitate to pronounce it white; and if a sheet of paper of a blue tint be placed by its side, it will appear to be white like the former.

Suppose again that we are in an apartment, the windows of which are completely exposed to the sun, and that they are closed by red curtains, the room will then be illumined entirely by red light; at the end of some moments the eye familiarised with the red tint spread over all the objects, will recognise as white those which are of this colour, and will regard also as white those which are of the red colour of the curtains. But this is not all: if, in the curtain an opening be made of about an eighth of an inch in diameter, and a piece of white paper be held at a little distance to receive the pencil of rays from the sun which passes through the opening, these rays will paint upon the paper a green spot; and if the curtains were green the spot would be red.

The reason why the spot is green in the first case and red in the second cannot be here explained, because this phenomenon depends upon the theory of the composition of light; but we proceed to attempt an explanation of how it happens that, the apartment being illumined by the red light, a white object which receives this light appears still white, a red object appears equally white, and why the white light of the solar rays, which experiences no alteration, since it passes through the hole in the curtain, and is received upon white paper, yet appears of a totally different colour.

It will be necessary to precede what we have to say upon this subject by some reflections upon the office generally performed by the white light in the operation of vision.

When looking at a body, no matter of what colour, each molecule of its visible surface remits to the eye white rays, together with those which are impressed with the colour peculiar to the body.

The more rays of this kind there are remitted, the more highly illumined the object appears, or the more vivid and clear its colour. Look at cinnabar, a substance composed of sulphur and mercury, from which is obtained the brilliant red employed in painting upon glass; in the mass cinnabar is of a red brown, dull enough, and like that of a thoroughly burnt brick; but, as it is broken up, it loses this obscure and deep colour; when divided it acquires more surface, and remits white light from a greater number of points; at length, when reduced to an impalpable powder, it presents a very brilliant red, and becomes vermilion. Each molecule of the cinnabar remits then to the eye more or less white light, and it is when they can reflect it in the greatest quantity, that this substance takes a colour the most brilliant.

Again, examining a hat, each hair of which the felt is composed is a small cylinder, which, seen through a microscope, presents a white edge, like that which is seen on a stick of sealing-wax, when viewed in a strong light: this edge, then, remits to the eye white light. What has just been found to be the case in these two examples is true of every body in nature; it is this white light, reflected from every visible point, which essentially determines the degree of brightness belonging to each portion of the object considered, because these white rays are the most perfect and the most vivid of those remitted to us by each particle; it is these, consequently, which enable us best to recognise the form, to appreciate the inclination of each element, and the curvature at each point of the surface. We are accustomed to this great abundance of white light, and to the services rendered by it to vision; and it is by comparison with it that coloured light is usually judged.

This being premised, if the objects are illumined only with

light previously coloured, if, as supposed above, red curtains or windows give this colour to all the light that the sun throws into the apartment, the forms of bodies in the room will no longer be judged of by means of white light, since the white rays which each point would have reflected, if the light had remained unaltered, have become red. These rays, however, are still the most perfect and the most vivid that arrive at the eye, and, although the eye is affected by them in a different manner, it still judges by their assistance, as it would have judged by the aid of the white rays: it is, therefore, naturally led to regard them as white, and it is by comparing them with these that it appreciates the colours of other rays. It is seen from this that, if there be in the apartment a body of the same red colour as the light with which the room is illumined, this object, remitting rays of the same nature as those that we esteem white, appears to us equally white. This experiment may easily be verified by placing a red glass before the eyes, and viewing through it white and red objects; both will appear of the former colour.

The same cause which makes us consider as white, rays which are not so in fact, prevents us from admitting as such, those which really are so, and this is the reason why the natural light of the sun, which passes through the little opening in the red curtain, throws on the white paper a colour which appears to us very perceptibly different from white.

The preceding observations made upon a particular example, are of such a nature as to be easily generalised and extended to all cases of bodies illumined by light of a different character from that habitually received from the sun. It is perceived how essential it may be sometimes to have regard to them, especially when painting an object which receives only reflected light, or light modified by the transparent media traversed by it. Nearly always, light which arrives only by reflexion is imprinted with the colour of the bodies reflecting it; this modification influences the appearances presented by the colours of the objects illumined by it, and the judgment formed with respect to them.

CHAPTER V.

ISOMETRIC PROJECTION.

(Pl. XII. to XIV.)

102. Although, generally, two projections are necessary for the complete delineation of any object, it has already been explained that a perfect idea of the object may be given by a single projection with the assistance of the traces of the shadows thrown by the bodies upon the plane of projection, when the manner in which they are supposed to be illumined is given. This method, however, would require great accuracy in the construction of the shadows, and would frequently present great difficulties in practice.

By the method about to be described, the invention of the late Professor Farish, a single projection, the construction of which is remarkable for its simplicity, forms a conventional picture, conveying at once to the eye the actual appearance of objects, as in a perspective view, and also giving readily the dimensions of the objects represented, especially those dimensions which are situated in planes parallel to three principal planes at right angles to each other. Now in a great many pieces of machinery, and in most architectural constructions, the principal parts lie in planes at right angles to each other; to such objects, consequently the method is particularly applicable.

The principle of isometric projection consists in selecting for the plane of projection a plane equally inclined to three principal axes at right angles to each other, called the *Isometric Axes*, so that all straight lines coincident with or parallel to these axes are represented upon the picture to the same scale; and, as will be presently seen a variable scale may readily be constructed for the measurement of all distances in planes parallel to those which contain the isometric axes.

All lines parallel to the isometric axes are, consequently, called *Isometric Lines;* the planes containing the isometric axes, and all planes parallel to these, are called *Isometric Planes.* The point in the object projected, which is assumed as the origin of the axes, is called the *Regulating Point.*

Isometric Projection of a Cube.

103. Let H B (pl. XII., fig. 41) be a perspective representation of the cube of which the isometric projection is to be constructed. Let A be assumed for the regulating point, and A B, A C, A D for the isometric axes; then the plane of projection making equal angles with A B, A C, and A D, the three lower edges of the cube, will also make equal angles with the other edges which are parallel to these, and will be perpendicular to A F, the diagonal of the cube. The projection of the diagonal A F will therefore be the point A, and the projections of all the edges of the cube will be equal straight lines, since they are equally inclined to the plane of projection. Moreover A B, A C, and A D being similarly situated with respect to the plane of projection, and the angles between them being equal, the angles between their projections will be equal, and therefore, each of them will be a third of four right angles. Suppose, now, the cube and plane of projection to be turned round together, till the diagonal plane D G, coincides with the plane of the paper (fig. 42), then the intersection of the plane of projection with the plane of the paper will be the straight line M N at right angles to the diagonal A F; letting fall the perpendicular D M upon M N, A M is obtained for the projection both of the edge A D, of the cube, and of D F the diagonal of the face H E; and similarly A N, equal to A M, and in the same straight line with it, is the projection both of the edge F G, and of the diagonal A G. In like manner the projections of all the other diagonals proceeding from the points A and F are coincident with the projections of edges of the cube. Again the points B, C, and D (fig. 41) being equally distant from the plane of projection, and on the

same side of it, the diagonals B C, C D, D B, joining these points, are parallel to the plane of projection, and their projections are consequently equal to the diagonals themselves, as also, for a like reason, are the projections of the diagonals E G, G H, H E. Moreover, the angle opposite any one of these diagonals contained by two projected edges of the cube being 120° the angles which these projected edges make with the diagonal are each equal to 30°. Hence is derived the following simple method of constructing the required isometric projection. Construct the face F E D H of the cube (fig. 43), and draw its diagonal H E, from E and H draw E C and H B making angles of 30° each with H E, and intersecting in F; complete the rhombus F′ E′ D′ H′, and this will be the projection of the face F E D H; join D′ F′, and produce it to G′, making F′ G′ equal to F′ D′, F′ E′, or F′ H′, and completing the rhombuses E′ G′, G′ H′, these will be the projections of the two other upper faces of the cube. The projections of the lower edges of the cube are coincident with the diagonals of the upper faces, and *vice versâ;* the rhombuses D C, C B, B D, are, therefore the projections of the lower faces; and the isometric projection of the cube is completely constructed.

As it may be frequently necessary to represent lines in different planes on the same portion of the picture, the different isometric planes may be distinguished by a difference of shading (fig. 44), and lines in other planes may be distinguished by representing the shadows thrown by them upon the isometric planes.

If the length of an edge, A D, of the cube (fig. 42) be represented by 1, D F, the diagonal of a face is represented by $\sqrt{2}$ and A F, the diagonal of the cube, by $\sqrt{3}$; and, since A F : F D : : A D : A M, the isometric projection, A M, of an edge of the cube is equal to $\dfrac{1 \times \sqrt{2}}{\sqrt{3}} = \frac{1}{3}\sqrt{6} = \cdot 8164966$. Those projections of diagonals of faces of the cube, which proceed from the regulating point, being each equal to the projection of an edge, are also equal to $\frac{1}{3}\sqrt{6}$, while the projections of the remaining diagonals are each equal to $\sqrt{2}$.

104. Hitherto the projection is a representation of the object in its full dimensions, but, if it be required to represent the object on a reduced scale, either one or the other of the following methods may be adopted. The dimensions of the object may be considered as first reduced to a given scale, a scale of $\frac{1}{30}$ for instance, and these reduced dimensions may then be isometrically projected: in this case the drawing represents the projection of a model of the object made to a reduced scale of $\frac{1}{30}$; and the dimensions in the directions of the isometric axes are on a scale of $\frac{1}{30} \cdot \frac{1}{3} \sqrt{6}$. The second method consists in setting off at once along the isometric axes, upon a given reduced scale, a scale of $\frac{1}{30}$ for instance, the dimensions to be represented in the directions of those axes; in this case the drawing represents the isometric projection of a model of the object made to a scale of $\frac{1}{30} \cdot \frac{1}{2} \sqrt{6}$. The latter method has this advantage, that distances along all isometric lines can be set off at once by means of the ordinary plotting scales, or by the Marquois's scales.

If the first of the above methods is adopted a scale for the dimensions in the directions of the isometric axes, may readily be constructed as follows. Referring to fig. 43, it is seen that the line D E and its isometric projection, containing an angle of 15°, form two sides of a triangle, the remaining side of which makes with D E an angle of 45°. Construct, therefore, the given scale, $\frac{1}{30}$ for instance, upon the line A B (pl. XIII., fig. 45), make the angles B A C and A B C equal to 15° and 45°, respectively, and drawing through the divisions upon A B, lines parallel to B C, the intersections of these lines with A C will form the isometric scale required. On the other hand, if the given scale be constructed upon A C, and lines parallel to B C be drawn to intersect A B, the scale upon A B will be that of the model, of which the second method gives the isometrical projection.

105. If A′B′, A′C′, A′D′ (pl. XII., fig. 43), the three lower edges of the cube be produced indefinitely, we have the projections of three axes at right angles to each other with reference, to which the position of any point being given, its isometric pro-

jection, may be determined; or conversely, from its isometric projection, its position in space may be estimated. Thus let the distances along the rectangular axes, which determine the position of a point, be 35, 40, and 60; and let A X, A Y, and A Z (pl. XIII., fig. 46), be the isometric axes; then setting off from a given scale the distances, A M = 35, A N = 40, and A O = 60, and completing the construction in the figure, P is the projection of the given point. Conversely the projection P being given, the distances A M, A N, and A O, estimated upon the isometric scale, determine the position of the point P in space, with reference to the three rectangular planes of which X Y, Z X, and Z Y are the projections.

One of the isometric planes, as X Y is generally considered to represent the horizontal plane, and then the other two, Z X and Z Y represent vertical planes.

To represent straight lines, not in the isometric directions, the isometric projections of their extremities must be found by the method just given, and the lines joining them will be the projections of the given lines.

To represent a curved line, the projections of a sufficient number of points in them must be determined, and the curve passing through these points will be the projection of the given curve.

If any point in the same isometric plane with a point to be projected, be already represented in the picture, that point may be assumed as a new regulating point, and the required point be formed by taking two distances only along the two new isometric axes lying in the given isometric plane; and if any point in the same isometric line with a point to be projected, be already represented, the required projection may be found by taking one distance only from that point along the given isometric line.

106. To construct scales for measuring all distances and angles lying in the isometric planes.

If circles be inscribed in the squares which form the faces of

a cube, the projections of these circles will be ellipses all of the same form and magnitude (pl. XIII., fig. 47). In each of these the projection of that diameter, which is parallel to the plane of projection, is the greatest, and is in fact equal to the diameter of the circle, and, consequently, to the edge of the cube. This projection is in the direction of the diagonal of the face, which is parallel to the plane of projection. The projection of the diameter at right angles to this, and which is in the direction of the inclined diagonal is the least. The former is therefore the major axis of the ellipse into which the circle is projected, and the latter is its minor axis.

The diameters of the ellipse, which are the projections of diameters of the circle, parallel to the edges of the cube, and are therefore parallel to the isometric axes, are called isometric diameters.

Now, if the extremities of the diameters of the inscribed circle, which lie in the directions of the diagonals of the face, be joined, a new square will be formed having its sides parallel to the sides of the face; their projections, therefore, will also be parallel to the projections of the sides of the face; and hence the ellipse may be easily constructed. Drawing, in fact, the diagonals F′ D′ and H′ E′, and setting off O M and O M′, each equal to half the edge of the cube, M O M′ is the major axis; and drawing M N and M′ N′ parallel to E′ D′ and H′ F′, and consequently, inclined to M M′ at angles of 30°; N O N′ is the minor axis of the ellipse. If it be required to make a scale for an isometric projection already constructed, the length of the edge of the cube must first be found by drawing E′ D inclined to H′ E′ at an angle of 45°.

Now to form the desired scale, we have only to describe a series of these ellipses, making the intervals between the ellipses in the isometric directions from the centre equal to the units on the isometric scale, or the intervals in the direction of the major axis equal to the units of the natural scale, or scale of the model of which the picture is the projection; the intersections of these

ellipses with any other line drawn through the center will, then, be at intervals, which are the units on that line, and will therefore form the scale for that line, and all lines having the same position with respect to the isometric lines. If three such elliptic scales be formed (pl. XIV., fig. 48), one for each principal isometric plane, having its isometric diameters parallel to the isometric axes in this plane, the scale for any line in this plane, or any plane parallel to it, will be found by drawing a diameter parallel to the line through the corresponding elliptic scale.

To graduate the elliptical scale so as to form a protractor, for measuring or setting off angles upon the corresponding isometric planes, the circle described upon the major axis of the ellipse must first be graduated, and the projections of these graduations must then be found upon the circumference of the ellipse. Thus, let M M′ (pl. XIII., fig. 49) be the axis major of the ellipse, which represents the isometric projection of the circle of which M M′ is the diameter. Let this circle be described, and graduated at the points *a*, *b*, *c*, &c.; then since the diameter M M′ is parallel to the plane of projection, the projections of the points *a*, *b*, *c*, &c., will be in planes perpendicular to M M′ and passing through *a*, *b*, *c*, &c.; and consequently will be the intersections, *a′*, *b′*, *c′*, &c., with the circumference of the ellipse of the perpendiculars let fall from *a*, *b*, *c*, &c., upon M M′. The three elliptical scales being graduated after this manner, the angles between any two lines in an isometric plane, will be the same as that indicated on the corresponding elliptical scale between two diameters parallel to those lines.

107. It frequently happens, for instance in representing wheels in machinery, that the projections of circles in isometric planes have to be drawn; and after what has been stated above, these projections may easily be constructed from given dimensions and positions of the circles, or the magnitudes and positions of the circles may be known at once from their projections. If these circles, as in the case of wheels, have axles through their centers at right angles to their planes, these axles being perpendicular

to the isometric planes, must be in the direction of the minor axes of the ellipses.

Lastly, with respect to the relative numerical magnitudes of the principal lines in the ellipses, referring again to the ellipse, which is the projection of a circle inscribed in a face of the cube, the major axis of this ellipse is equal to an edge of the cube, and its isometric diameter is equal to the projection of this edge: the ratio of the major axis to the isometric diameter is therefore $1 : \frac{1}{3}\sqrt{6}$. Again, the projections of the diameters lying upon the diagonals of the face, and, consequently, being to each other as the projections of the diagonals, the major axis of the ellipse has to its minor axis the ratio $\sqrt{2} : \frac{1}{3}\sqrt{6}$ or $1 : \frac{1}{3}\sqrt{3}$. Hence the minor axis, the isometric diameter, and the major axis have the ratios $\frac{1}{3}\sqrt{3} : \frac{1}{3}\sqrt{6} : 1$.

The application of the preceding principles is illustrated by fig. 48, which represents the isometric projection of a stand for a telescope, accompanied by elliptical scales of feet and inches for each of the isometric planes, and graduations for measuring angles in these planes.

W. OSTELL, PRINTER, HART STREET, BLOOMSBURY.

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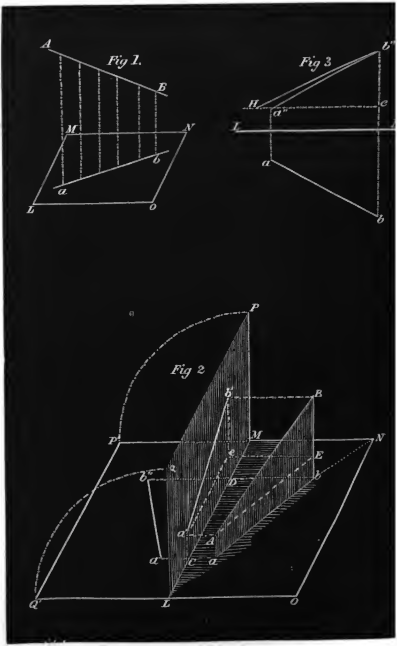

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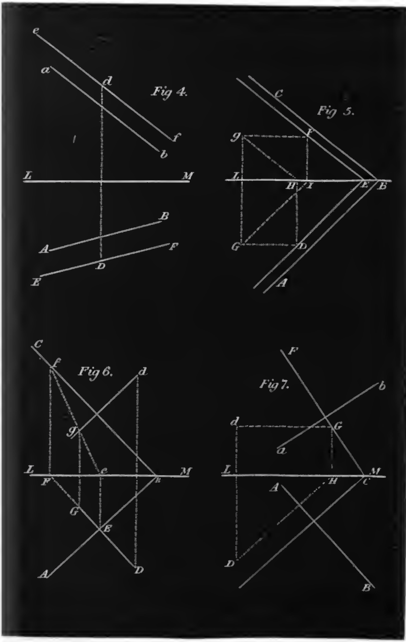
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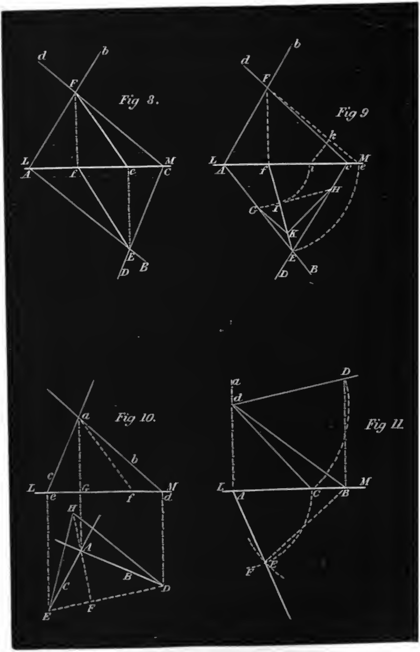
Fig 8.
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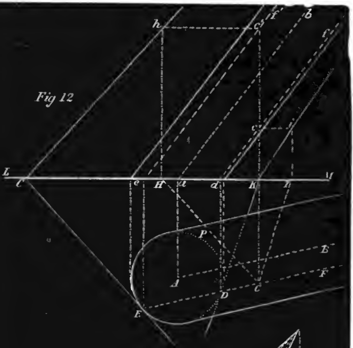
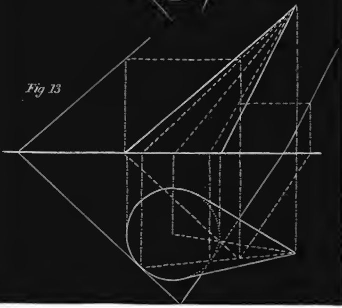


Fig 13



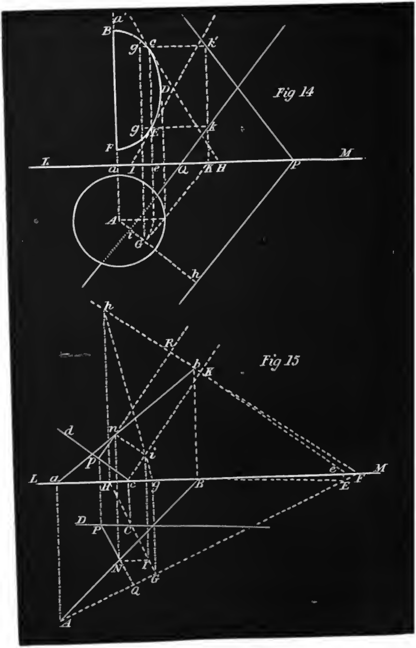


Fig 14
Fig 15

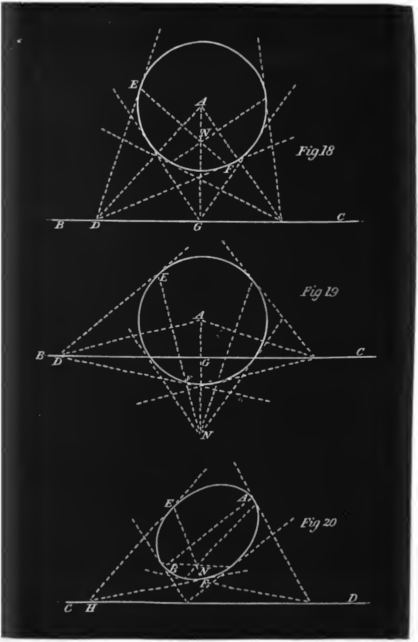


Fig 18

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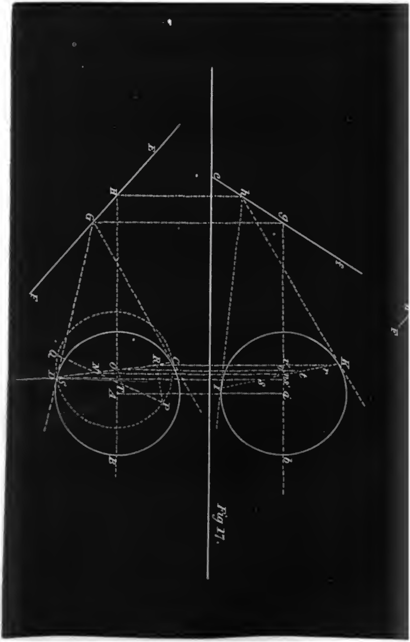

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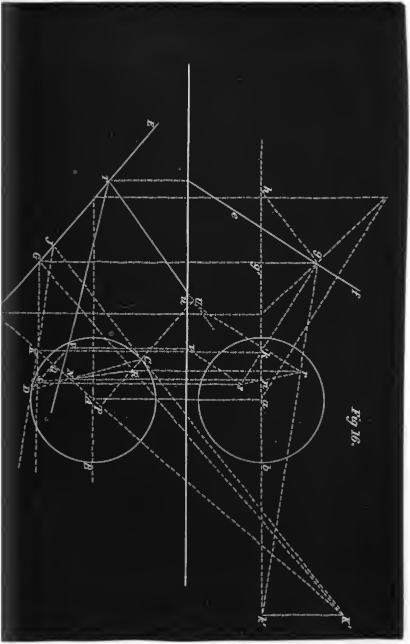

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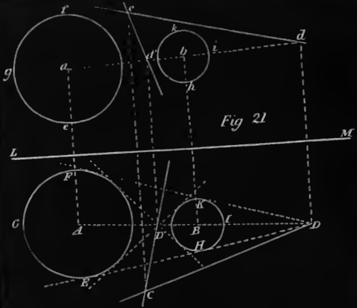
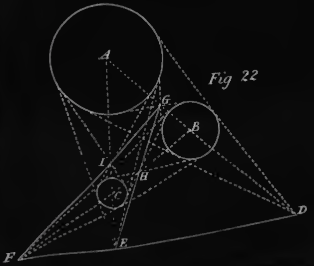


Fig 21



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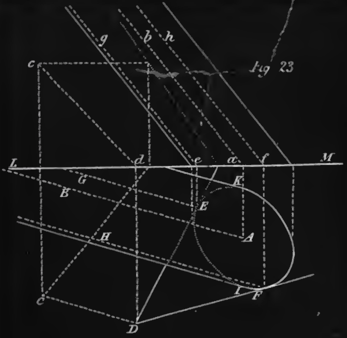


Fig 23

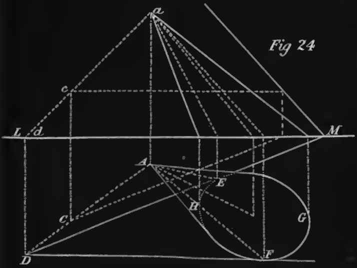


Fig 24

PLATE

VI.

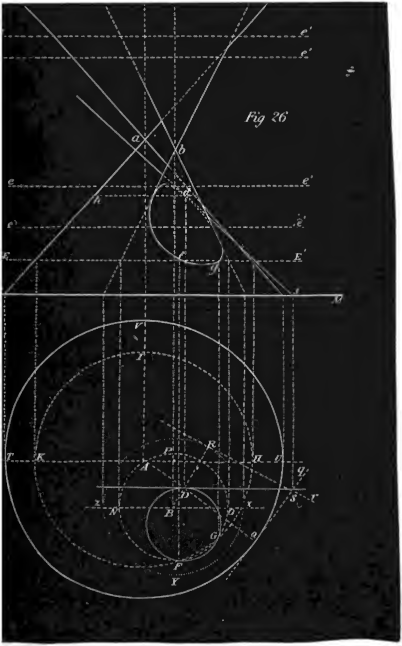


Fig 26

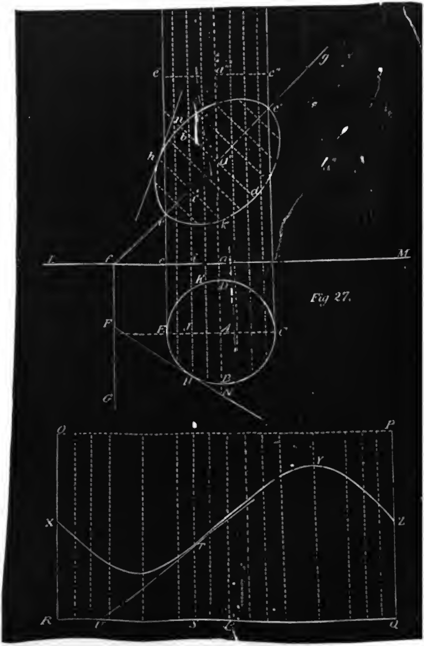
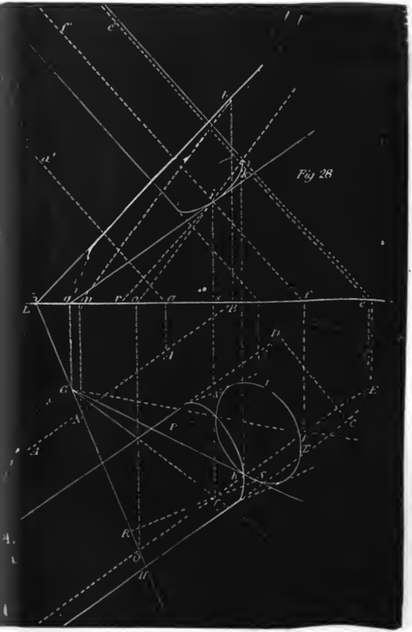


Fig 27.

TE VII.



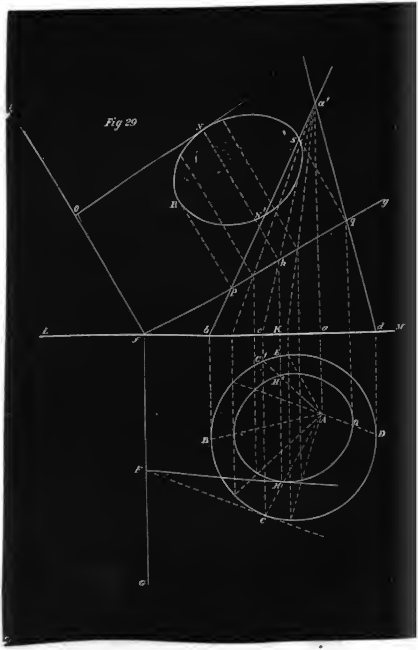
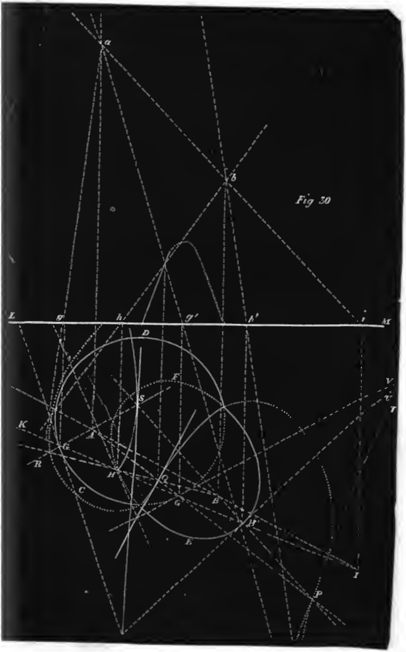


Fig 29

VIII.



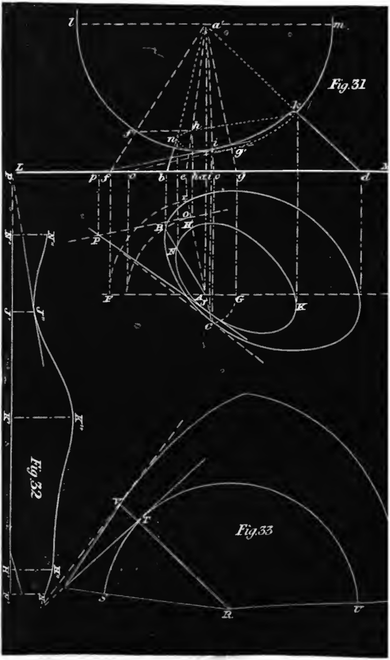


Fig. 31

Fig. 32

Fig. 33

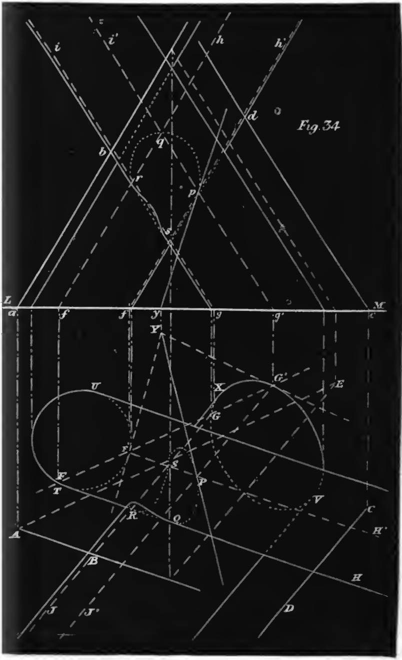


Fig. 34
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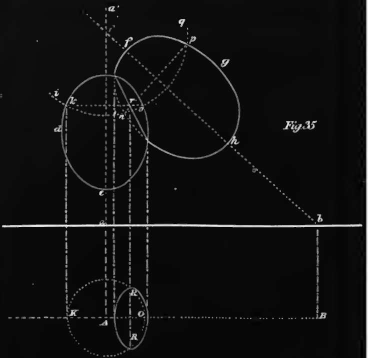


Fig.35

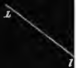
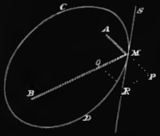


Fig 36



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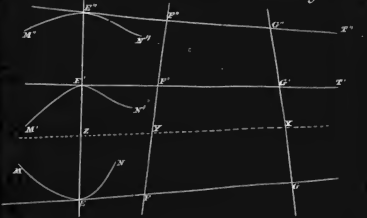


Fig 37

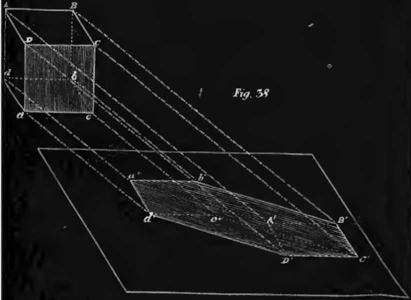


Fig. 38

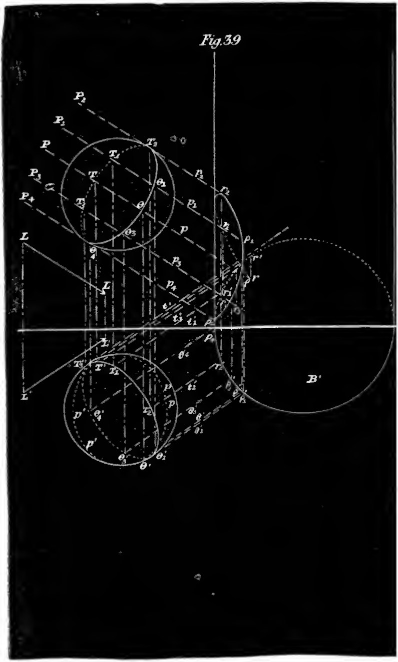


Fig. 39

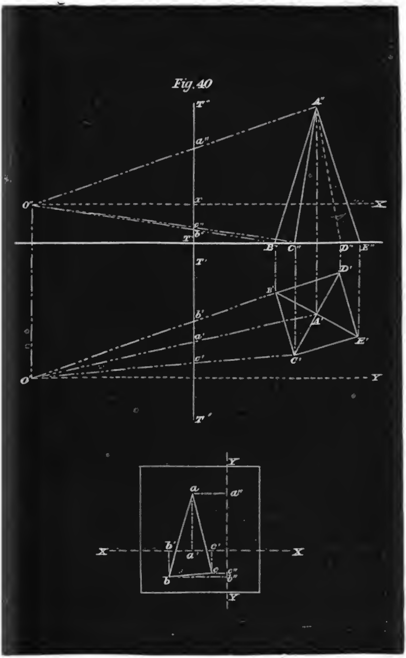


Fig. 40

Fig. 41

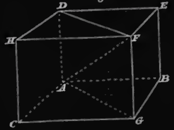
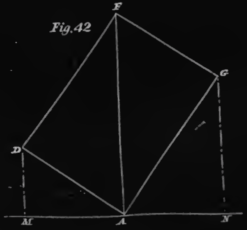


Fig. 42



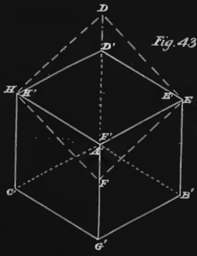
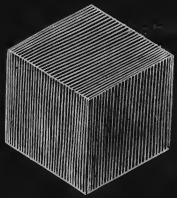


Fig. 43

Fig. 44



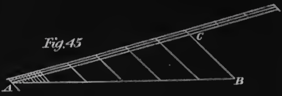


Fig.45

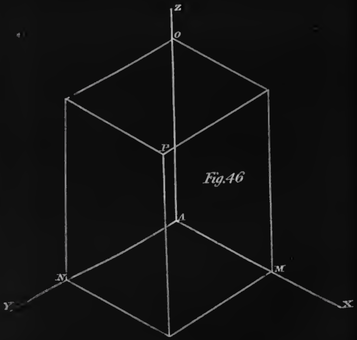


Fig.46

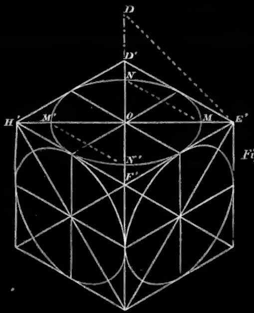


Fig. 47

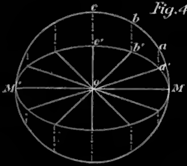


Fig. 49

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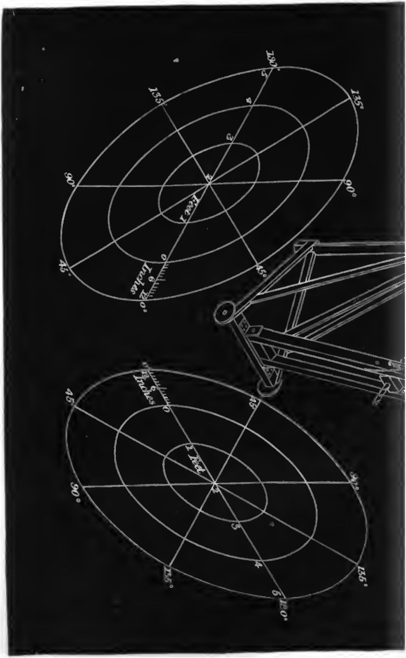

ATE XIV.

Fig. 48

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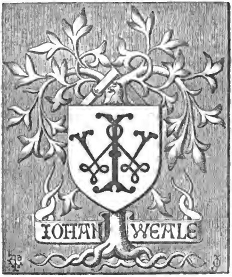

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4